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# AN INTRODUCTION TO SEMILINEAR ELLIPTIC EQUATIONS

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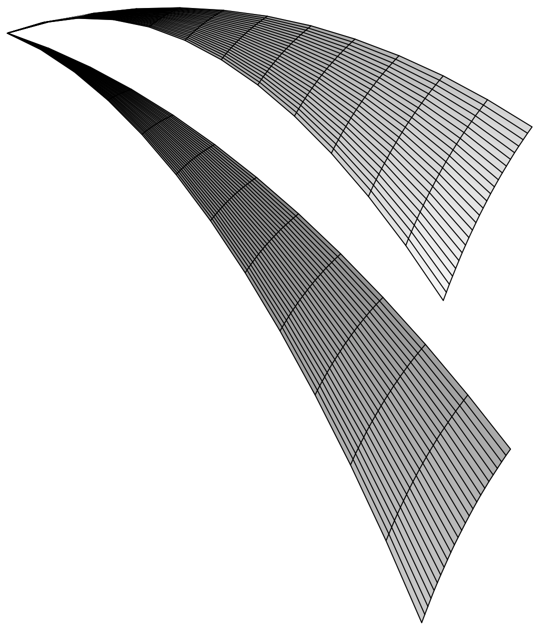

# An introduction to semilinear elliptic equations

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# Contents

# Introduction

These notes contain the material of a course given at the Institute of Mathematics of the Federal University of Rio de Janeiro during the second semester of 1996. The aim of these notes is to present a few methods that are useful for the study of nonlinear partial differential equations of elliptic type. Every method which is introduced is illustrated by specific examples, describing various properties of elliptic equations.

The reader is supposed to be familiar with the basic properties of ordinary differential equations, with elementary functional analysis and with the elementary theory of integration, including $L^p$ spaces. Of course, we use Sobolev spaces in most of the chapters, and so we give a self-contained introduction to those spaces (containing all the properties that we use) in an appendix at the end of the notes.

We study the model problem

$$\begin{cases} -\triangle u = g & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega. \end{cases}$$

Here, $g = g(x, u)$ is a function of $x \in \Omega$ and $u \in \mathbb{R}$, and $\Omega$ is an open domain of $\mathbb{R}^N$. This is clearly not the most general elliptic problem, but we simply whish to introduce some basic tools, so we leave to the reader the possible adaptation of the methods to more general equations and boundary conditions.

The first chapter is devoted to ODE methods. We first study the one dimensional case, and give a complete description of the solutions. We next study the higher dimensional problem, when $\Omega$ is a ball or the whole space, by the shooting method.

In the second chapter, we first study the linear equation, and then we present some variational methods: global and constrained minimization and the mountain pass theorem. We also introduce two techniques that can be used to handle the case of unbounded domains, symmetrization and concentration-compactness.

The third chapter is devoted to the method of super- and subsolutions. We first introduce the weak and strong maximum principles, and then an existence result based on an iteration technique.

In the fourth chapter, we study some qualitative properties of the solutions. We study the $L^p$ and $C_0$ regularity for the linear equation, and then the regularity for nonlinear equations by a bootstrap method. Finally, we study the symmetry properties of the solutions by the moving planes technique.

Of course, there are other important methods for the study of elliptic equations, in particular the degree theory and the bifurcation theory. We did not study these methods because their most interesting applications require the use of the $C^{m,\alpha}$ regularity theory, which we could not afford to present in such an introductory text. The interested reader might consult for example H. Brezis and L. Nirenberg [**14**].

# Notation

| | |
|---|---|
| a.a. | almost all |
| a.e. | almost everywhere |
| $\overline{E}$ | the closure of the subset $E$ of the topological space $X$ |
| $C^k(E,F)$ | the space of $k$ times continuously differentiable functions from the topological space $E$ to the topological space $F$ |
| $\mathcal{L}(E,F)$ | the Banach space of linear, continuous operators from the Banach space $E$ to the Banach space $F$, equipped with the norm topology |
| $\mathcal{L}(E)$ | the space $\mathcal{L}(E,E)$ |
| $X^\star$ | the topological dual of the space $X$ |
| $X \hookrightarrow Y$ | if $X \subset Y$ with continuous injection |
| $\Omega$ | an open subset of $\mathbb{R}^N$ |
| $\overline{\Omega}$ | the closure of $\Omega$ in $\mathbb{R}^N$ |
| $\partial\Omega$ | the boundary of $\Omega$, i.e. $\partial\Omega = \overline{\Omega} \setminus \Omega$ |
| $\omega \subset\subset \Omega$ | if $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ is compact |
| $\partial_i u$ | $= u_{x_i} = \dfrac{\partial u}{\partial x_i}$ |
| $\partial_r u$ | $= u_r = \dfrac{\partial u}{\partial r} = \dfrac{1}{r} x \cdot \nabla u$, where $r = \vert x\vert$ |
| $D^\alpha$ | $= \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}$ for $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ |
| $\nabla u$ | $(\partial_1 u, \cdots, \partial_N u)$ |
| $\triangle$ | $= \displaystyle\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ |
| $u \star v$ | the convolution in $\mathbb{R}^N$, i.e. $u \star v(x) = \displaystyle\int_{\mathbb{R}^N} u(y)v(x-y)\,dy = \int_{\mathbb{R}^N} u(x-y)v(y)\,dy$ |
| $\mathcal{F}$ | the Fourier transform in $\mathbb{R}^N$, defined by[1] $\mathcal{F}u(\xi) = \displaystyle\int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} u(x)\,dx$ |
| $\overline{\mathcal{F}}$ | $= \mathcal{F}^{-1}$, given by $\overline{\mathcal{F}}v(x) = \displaystyle\int_{\mathbb{R}^N} e^{2\pi i \xi\cdot x} v(\xi)\,d\xi$ |
| $\widehat{u}$ | $= \mathcal{F}u$ |
| $C_c(\Omega)$ | the space of continuous functions $\Omega \to \mathbb{R}$ with compact support |
| $C_c^k(\Omega)$ | the space of functions of $C^k(\Omega)$ with compact support |
| $C_b(\Omega)$ | the Banach space of continuous, bounded functions $\Omega \to \mathbb{R}$, equipped with the topology of uniform convergence |
| $C(\overline{\Omega})$ | the space of continuous functions $\overline{\Omega} \to \mathbb{R}$. When $\Omega$ is bounded, $C(\overline{\Omega})$ is a Banach space when equipped with the topology of uniform convergence |

[1] with this definition of the Fourier transform, $\|\mathcal{F}u\|_{L^2} = \|u\|_{L^2}$, $\mathcal{F}(u \star v) = \mathcal{F}u\mathcal{F}v$ and $\mathcal{F}(D^\alpha u) = (2\pi i)^{|\alpha|} \prod_{j=1}^N x_j^{\alpha_j} \mathcal{F}u$.

| | |
|---|---|
| $C_{\mathrm{b,u}}(\overline{\Omega})$ | the Banach space of uniformly continuous and bounded functions $\overline{\Omega} \to \mathbb{R}$ equipped with the topology of uniform convergence |
| $C^m_{\mathrm{b,u}}(\overline{\Omega})$ | the Banach space of functions $u \in C_{\mathrm{b,u}}(\overline{\Omega})$ such that $D^\alpha u \in C_{\mathrm{b,u}}(\overline{\Omega})$, for every multi-index $\alpha$ such that $\lvert\alpha\rvert \le m$. $C^m_{\mathrm{b,u}}(\overline{\Omega})$ is equipped with the norm of $W^{m,\infty}(\Omega)$ |
| $C^{m,\alpha}(\overline{\Omega})$ | for $0 \le \alpha < 1$, the Banach space of functions $u \in C^m_{\mathrm{b,u}}(\overline{\Omega})$ such that $\|u\|_{C^{m,\alpha}} = \|u\|_{W^{m,\infty}} + \sup_{\substack{x,y\in\Omega \\ \lvert\beta\rvert=m}} \frac{\lvert D^\beta u(x) - D^\beta u(y)\rvert}{\lvert x-y\rvert^\alpha} < \infty.$ |
| $\mathcal{D}(\Omega)$ | $= C^\infty_{\mathrm{c}}(\Omega)$, the Fréchet space of $C^\infty$ functions $\Omega \to \mathbb{R}$ (or $\Omega \to \mathbb{C}$) compactly supported in $\Omega$, equipped with the topology of uniform convergence of all derivatives on compact subsets of $\Omega$ |
| $C_0(\Omega)$ | the closure of $C^\infty_{\mathrm{c}}(\Omega)$ in $L^\infty(\Omega)$ |
| $C^m_0(\Omega)$ | the closure of $C^\infty_{\mathrm{c}}(\Omega)$ in $W^{m,\infty}(\Omega)$ |
| $\mathcal{D}'(\Omega)$ | the space of distributions on $\Omega$, that is the topological dual of $\mathcal{D}(\Omega)$ |
| $p'$ | the conjugate of $p$ given by $\frac{1}{p} + \frac{1}{p'} = 1$ |
| $L^p(\Omega)$ | the Banach space of (classes of) measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega \lvert u(x)\rvert^p \,dx < \infty$ if $1 \le p < \infty$, or $\operatorname{ess\,sup}_{x\in\Omega} \lvert u(x)\rvert < \infty$ if $p = \infty$. $L^p(\Omega)$ is equipped with the norm $\|u\|_{L^p} = \begin{cases} \left(\int_\Omega \lvert u(x)\rvert^p \,dx\right)^{\frac{1}{p}} & \text{if } p < \infty \\ \operatorname{ess\,sup}_{x\in\Omega} \lvert u(x)\rvert, & \text{if } p = \infty. \end{cases}$ |
| $L^p_{\mathrm{loc}}(\Omega)$ | the set of measurable functions $u : \Omega \to \mathbb{R}$ such that $u_{\lvert\omega} \in L^p(\omega)$ for all $\omega \subset\subset \Omega$ |
| $W^{m,p}(\Omega)$ | the space of (classes of) measurable functions $u : \Omega \to \mathbb{R}$ such that $D^\alpha u \in L^p(\Omega)$ in the sense of distributions, for every multi-index $\alpha \in \mathbb{N}^N$ with $\lvert\alpha\rvert \le m$. $W^{m,p}(\Omega)$ is a Banach space when equipped with the norm $\|u\|_{W^{m,p}} = \sum_{\lvert\alpha\rvert\le m} \|D^\alpha u\|_{L^p}$ |
| $W^{m,p}_{\mathrm{loc}}(\Omega)$ | the set of measurable functions $u : \Omega \to \mathbb{R}$ such that $u_{\lvert\omega} \in W^{m,p}(\omega)$ for all $\omega \subset\subset \Omega$ |
| $W^{m,p}_0(\Omega)$ | the closure of $C^\infty_{\mathrm{c}}(\Omega)$ in $W^{m,p}(\Omega)$ |
| $W^{-m,p'}(\Omega)$ | the topological dual of $W^{m,p}_0(\Omega)$ |
| $H^m(\Omega)$ | $= W^{m,2}(\Omega)$. $H^m(\Omega)$ is equipped with the equivalent norm $\|u\|_{H^m} = \Big(\sum_{\lvert\alpha\rvert\le m} \int_\Omega \lvert D^\alpha u(x)\rvert^2 \,dx\Big)^{\frac{1}{2}},$ and $H^m(\Omega)$ is a Hilbert space for the scalar product $(u,v)_{H^m} = \int_\Omega \Re(u(x)\overline{v(x)})\,dx$ |
| $H^m_{\mathrm{loc}}(\Omega)$ | $= W^{m,2}_{\mathrm{loc}}(\Omega)$ |
| $H^m_0(\Omega)$ | $= W^{m,2}_0(\Omega)$ |
| $H^{-m}(\Omega)$ | $= W^{-m,2}(\Omega)$ |
| $\lvert u\rvert_{m,p,\Omega}$ | $= \sum_{\lvert\alpha\rvert=m} \|D^\alpha u\|_{L^p(\Omega)}$ |

CHAPTER 1

# ODE methods

Consider the problem

$$\begin{cases} -\triangle u = g(u) \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \partial\Omega, \end{cases}$$

where $\Omega$ is the ball

$$\Omega = \{x \in \mathbb{R}^N; |x| < R\},$$

for some given $0 < R \le \infty$. In the case $R = \infty$, the boundary condition is understood as $u(x) \to 0$ as $|x| \to \infty$. Throughout this chapter, we assume that $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function. We look for *nontrivial* solutions, i.e. solutions $u \not\equiv 0$ (clearly, $u \equiv 0$ is a solution if and only if $g(0) = 0$). In this chapter, we study their existence by purely ODE methods.

If $N = 1$, then the equation is simply the ordinary differential equation

$$u'' + g(u) = 0, \quad -R < r < R,$$

and the boundary condition becomes $u(\pm R) = 0$, or $u(r) \to 0$ as $r \to \pm\infty$ in the case $R = \infty$. In Sections 1.1 and 1.2, we solve completely the above problem. We give necessary and sufficient conditions on $g$ so that there exists a solution, and we characterize all the solutions.

In the case $N \ge 2$, then one can also reduce the problem to an ordinary differential equation. Indeed, if we look for a radially symmetric solution $u(x) = u(|x|)$, then the equation becomes the ODE

$$u'' + \frac{N-1}{r}u' + g(u) = 0, \quad 0 < r < R,$$

and the boundary condition becomes $u(R) = 0$, or $u(r) \to 0$ as $r \to \infty$ in the case $R = \infty$. The approach that we will use for solving this problem is the following. Given $u_0 > 0$, we solve the ordinary differential equation with the initial values $u(0) = u_0$, $u'(0) = 0$. There exists a unique solution, which is defined on a maximal interval $[0, R_0)$. Next, we try to *adjust* the initial value $u_0$ in such a way that $R_0 > R$ and $u(R) = 0$ ($R_0 = \infty$ and $\lim_{r\to\infty} u(r) = 0$ in the case $R = \infty$). This is called the *shooting* method. In Sections 1.3 and 1.4, we give sufficient conditions on $g$ for the existence of solutions. We also obtain some necessary conditions.

## 1.1. The case of the line

We begin with the simple case $N = 1$ and $R = \infty$. In other words, $\Omega = \mathbb{R}$. In this case, we do not need to impose radial symmetry (but we will see that any solution is radially symmetric up to a translation). We consider the equation

$$u'' + g(u) = 0, \tag{1.1.1}$$

for all $x \in \mathbb{R}$, with the boundary condition

$$\lim_{x\to\pm\infty} u(x) = 0. \tag{1.1.2}$$

We give a necessary and sufficient condition on $g$ for the existence of nontrivial solutions of (1.1.1)–(1.1.2). Moreover, we characterize all solutions. We show that all solutions are derived from a unique positive, even one and a unique negative, even one (whenever they exist) by translations.

We begin by recalling some elementary properties of the equation (1.1.1).

REMARK 1.1.1. The following properties hold.

(i) Given $x_0, u_0, v_0 \in \mathbb{R}$, there exists a unique solution $u$ of (1.1.1) such that $u(x_0) = u_0$ and $u'(x_0) = v_0$, defined on a maximal interval $(a, b)$ for some $-\infty \le a < x_0 < b \le \infty$. In addition, if $a > -\infty$, then $|u(x)| + |u'(x)| \to \infty$ as $x \downarrow a$ (similarly, $|u(x)| + |u'(x)| \to \infty$ as $x \uparrow b$ if $b < \infty$). This is easily proved by solving the integral equation
$$u(x) = u_0 + (x - x_0)v_0 - \int_{x_0}^{x}\int_{x_0}^{s} g(u(\sigma))\, d\sigma\, ds,$$
on the interval $(x_0 - \alpha, x_0 + \alpha)$ for some $\alpha > 0$ sufficiently small (apply Banach's fixed point theorem in $C([x_0 - \alpha, x_0 + \alpha])$), and then by considering the maximal solution.

(ii) It follows in particular from uniqueness that if $u$ satisfies (1.1.1) on some interval $(a, b)$ and if $u'(x_0) = 0$ and $g(u(x_0)) = 0$ for some $x_0 \in (a, b)$, then $u \equiv u(x_0)$ on $(a, b)$.

(iii) If $u$ satisfies (1.1.1) on some interval $(a, b)$ and $x_0 \in (a, b)$, then
$$\frac{1}{2}u'(x)^2 + G(u(x)) = \frac{1}{2}u'(x_0)^2 + G(u(x_0)), \tag{1.1.3}$$
for all $x \in (a, b)$, where
$$G(s) = \int_0^s g(\sigma)\, d\sigma, \tag{1.1.4}$$
for $s \in \mathbb{R}$. Indeed, multiplying the equation by $u'$, we obtain
$$\frac{d}{dx}\Big\{\frac{1}{2}u'(x)^2 + G(u(x))\Big\} = 0,$$
for all $x \in (a, b)$.

(iv) Let $x_0 \in \mathbb{R}$ and $h > 0$. If $u$ satisfies (1.1.1) on $(x_0 - h, x_0 + h)$ and $u'(x_0) = 0$, then $u$ is symmetric about $x_0$, i.e. $u(x_0 + s) \equiv u(x_0 - s)$ for all $0 \le s < h$. Indeed, let $v(s) = u(x_0 + s)$ and $w(s) = u(x_0 - s)$ for $0 \le s < h$. Both $v$ and $w$ satisfy (1.1.1) on $(-h, h)$ and we have $v(0) = w(0)$ and $v'(0) = w'(0)$, so that by uniqueness $v \equiv w$.

(v) If $u$ satisfies (1.1.1) on some interval $(a, b)$ and $u'$ has at least two distinct zeroes $x_0, x_1 \in (a, b)$, then $u$ exists on $(-\infty, +\infty)$ and $u$ is periodic with period $2|x_0 - x_1|$. This follows easily from (iv), since $u$ is symmetric about both $x_0$ and $x_1$.

We next give some properties of possible solutions of (1.1.1)–(1.1.2).

LEMMA 1.1.2. *If $u \not\equiv 0$ satisfies (1.1.1)–(1.1.2), then the following properties hold.*

(i) $g(0) = 0$.

(ii) *Either $u > 0$ on $\mathbb{R}$ or else $u < 0$ on $\mathbb{R}$.*

(iii) *$u$ is symmetric about some $x_0 \in \mathbb{R}$, and $u'(x) \ne 0$ for all $x \ne x_0$. In particular, $|u(x - x_0)|$ is symmetric about $0$, increasing for $x < 0$ and decreasing for $x > 0$.*

(iv) *For all $y \in \mathbb{R}$, $u(\cdot - y)$ satisfies (1.1.1)–(1.1.2).*

PROOF. If $g(0) \ne 0$, then $u''(x)$ has a nonzero limit as $x \to \pm\infty$, so that $u$ cannot have a finite limit. This proves (i). By (1.1.2), $u$ cannot be periodic.

Therefore, it follows from Remark 1.1.1 (v) and (iv) that $u'$ has exactly one zero on $\mathbb{R}$ and is symmetric about this zero. Properties (ii) and (iii) follow. Property (iv) is immediate. □

By Lemma 1.1.2, we need only study the even, positive or negative solutions (since any solution is a translation of an even positive or negative one), and we must assume $g(0) = 0$. Our main result of this section is the following.

THEOREM 1.1.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous with $g(0) = 0$. There exists a positive, even solution of (1.1.1)–(1.1.2) if and only if there exists $u_0 > 0$ such that*

$$g(u_0) > 0, \quad G(u_0) = 0 \quad \text{and} \quad G(u) < 0 \quad \text{for} \quad 0 < u < u_0, \tag{1.1.5}$$

*where $G$ is defined by (1.1.4). In addition, such a solution is unique. Similarly, there exists a negative, even solution of (1.1.1)–(1.1.2) if and only if there exists $v_0 < 0$ such that*

$$g(v_0) < 0, \quad G(v_0) = 0 \quad \text{and} \quad G(u) < 0 \quad \text{for} \quad v_0 < u < 0, \tag{1.1.6}$$

*and such a solution is unique.*

PROOF. We only prove the first statement, and we proceed in five steps.

STEP 1. Let $x_0 \in \mathbb{R}$ and let $u \in C^2([x_0, \infty))$. If $u(x) \to \ell \in \mathbb{R}$ and $u''(x) \to 0$ as $x \to \infty$, then $u'(x) \to 0$. Indeed, we have

$$u'(s) = u'(x) + \int_x^s u''(\sigma)\, d\sigma,$$

for $s > x \geq x_0$. Therefore,

$$u(x+1) - u(x) = \int_x^{x+1} u'(s)\, ds = u'(x) + \int_x^{x+1} \int_x^s u''(\sigma)\, d\sigma\, ds,$$

from which the conclusion follows immediately.

STEP 2. If $u$ is even and satifies (1.1.1)–(1.1.2), then

$$\frac{1}{2} u'(x)^2 + G(u(x)) = 0, \tag{1.1.7}$$

for all $x \in \mathbb{R}$ and

$$G(u(0)) = 0. \tag{1.1.8}$$

Indeed, letting $x_0 \to \infty$ in (1.1.3), and using Step 1 and (1.1.2), we obtain (1.1.7). (1.1.8) follows, since $u'(0) = 0$.

STEP 3. If $u$ is a positive, even solution of (1.1.1)–(1.1.2), then $g$ satisfies (1.1.5) with $u_0 = u(0)$. Indeed, we have $G(u_0) = 0$ by (1.1.8). Since $u'(x) \neq 0$ for $x \neq 0$ (by Lemma 1.1.2 (iii)), it follows from (1.1.7) that $G(u(x)) < 0$ for all $x \neq 0$, thus $G(u) < 0$ for all $0 < u < u_0$. Finally, since $u$ is decreasing for $x > 0$ we have $u'(x) \leq 0$ for all $x \geq 0$. This implies that $u''(0) \leq 0$, i.e. $g(u_0) \geq 0$. If $g(u_0) = 0$, then $u \equiv u_0$ by uniqueness, which is absurd by (1.1.2). Therefore, we must have $g(u_0) > 0$.

STEP 4. If $g$ satisfies (1.1.5), then the solution $u$ of (1.1.1) with the initial values $u(0) = u_0$ and $u'(0) = 0$ is even, decreasing for $x > 0$ and satisfies (1.1.2). Indeed, since $g(u_0) > 0$, we have $u''(0) < 0$. Thus $u'(x) < 0$ for $x > 0$ and small. $u'$ cannot vanish while $u$ remains positive, for otherwise we would have by (1.1.7) $G(u(x)) = 0$ for some $x$ such that $0 < u(x) < u_0$. This is ruled out by (1.1.5). Furthermore, $u$ cannot vanish in finite time, for then we would have $u(x) = 0$ for some $x > 0$ and thus $u'(x) = 0$ by (1.1.7), which would imply $u \equiv 0$ (see Remark 1.1.1 (ii)). Therefore, $u$ is positive and decreasing for $x > 0$, and thus has a limit $\ell \in [0, u_0)$ as $x \to \infty$. We show that $\ell = 0$. Since $u''(x) \to g(\ell)$ as $x \to \infty$, we must have $g(\ell) = 0$. By Step 1, we deduce that $u'(x) \to 0$ as $x \to \infty$. Letting $x \to \infty$ in (1.1.7) (which holds, because of (1.1.3) and the assumption $G(u_0) = 0$), we find $G(\ell) = 0$, thus $\ell = 0$. Finally, $u$ is even by Remark 1.1.1 (iv).

STEP 5. Conclusion. The necessity of condition (1.1.5) follows from Step 3, and the existence of a solution follows from Step 4. It thus remain to show uniqueness. Let $u$ and $\widetilde{u}$ be two positive, even solutions. We deduce from Step 3 that $g$ satisfies (1.1.5) with both $u_0 = u(0)$ and $u_0 = \widetilde{u}(0)$. It easily follows that $\widetilde{u}(0) = u(0)$, thus $\widetilde{u}(x) \equiv u(x)$. □

REMARK 1.1.4. If $g$ is odd, then the statement of Theorem 1.1.3 is simplified. There exists solution $u \not\equiv 0$ of (1.1.1)–(1.1.2) if and only if (1.1.5) holds. In this case, there exists a unique positive, even solution of (1.1.1)–(1.1.2), which is decreasing for $x > 0$. Any other solution $\widetilde{u}$ of (1.1.1)–(1.1.2) has the form $\widetilde{u}(x) = \varepsilon u(x - y)$ for $\varepsilon = \pm 1$ and $y \in \mathbb{R}$.

REMARK 1.1.5. Here are some applications of Theorem 1.1.3 and Remark 1.1.4.

(i) Suppose $g(u) = -\lambda u$ for some $\lambda \in \mathbb{R}$ (linear case). Then there is no nontrivial solution of (1.1.1)–(1.1.2). Indeed, neither (1.1.5) nor (1.1.6) hold. One can see this directly by calculating all solutions of the equation. If $\lambda = 0$, then all the solutions have the form $u(x) = a + bx$ for some $a, b \in \mathbb{R}$. If $\lambda > 0$, then all the solutions have the form $u(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}$ for some $a, b \in \mathbb{R}$. If $\lambda < 0$, then all the solutions have the form $u(x) = ae^{i\sqrt{-\lambda}x} + be^{-i\sqrt{-\lambda}x}$ for some $a, b \in \mathbb{R}$.

(ii) Suppose $g(u) = -\lambda u + \mu|u|^{p-1}u$ for some $\lambda, \mu \in \mathbb{R}$ and some $p > 1$. If $\lambda \le 0$ or if $\mu \le 0$, then there is no nontrivial solution of (1.1.1)–(1.1.2). If $\lambda, \mu > 0$, then there is the solution

$$u(x) = \Big(\frac{\lambda(p+1)}{2\mu}\Big)^{\frac{1}{p-1}} \Big(\cosh\Big(\frac{p-1}{2}\sqrt{\lambda}x\Big)\Big)^{-\frac{2}{p-1}}.$$

All other solutions have the form $\widetilde{u}(x) = \varepsilon u(x-y)$ for $\varepsilon = \pm 1$ and $y \in \mathbb{R}$. We need only apply Remark 1.1.4.

(iii) Suppose $g(u) = -\lambda u + \mu|u|^{p-1}u - \nu|u|^{q-1}u$ for some $\lambda, \mu, \nu \in \mathbb{R}$ and some $1 < p < q$. The situation is then much more complex.

a) If $\lambda < 0$, then there is no nontrivial solution.

b) If $\lambda = 0$, then the only case when there is a nontrivial solution is when $\mu < 0$ and $\nu < 0$. In this case, there is the even, positive decreasing solution $u$ corresponding to the intial value $u(0) = ((q+1)\mu/(p+1)\nu)^{\frac{1}{q-p}}$ and $u'(0) = 0$. All other solutions have the form $\widetilde{u}(x) = \varepsilon u(x - y)$ for $\varepsilon = \pm 1$ and $y \in \mathbb{R}$.

c) If $\lambda > 0$, $\mu \le 0$ and $\nu \ge 0$, then there is no nontrivial solution.

d) If $\lambda > 0$, $\mu > 0$ and $\nu \le 0$, then there is the even, positive decreasing solution $u$ corresponding to the intial value $u_0 > 0$ given by

$$\frac{\mu}{p+1}u_0^{p-1} - \frac{\nu}{q+1}u_0^{q-1} = \frac{\lambda}{2}. \tag{1.1.9}$$

All other solutions have the form $\widetilde{u}(x) = \varepsilon u(x - y)$ for $\varepsilon = \pm 1$ and $y \in \mathbb{R}$.

e) If $\lambda > 0$, $\mu > 0$ and $\nu > 0$, let $\overline{u} = ((q+1)(p-1)\mu/(p+1)(q-1)\nu)^{\frac{1}{q-p}}$. If

$$\frac{\mu}{p+1}\overline{u}^{p-1} - \frac{\nu}{q+1}\overline{u}^{q-1} \le \frac{\lambda}{2},$$

then there is no nontrivial solution. If

$$\frac{\mu}{p+1}\overline{u}^{p-1} - \frac{\nu}{q+1}\overline{u}^{q-1} > \frac{\lambda}{2},$$

then there is the even, positive decreasing solution $u$ corresponding to the intial value $u \in (0, \overline{u})$ given by (1.1.9). All other solutions have the form $\widetilde{u}(x) = \varepsilon u(x-y)$ for $\varepsilon = \pm 1$ and $y \in \mathbb{R}$.

f) If $\lambda > 0$ and $\nu < 0$, then there is the even, positive decreasing solution $u$ corresponding to the intial value $u_0 > 0$ given by (1.1.9). All other solutions have the form $\widetilde{u}(x) = \varepsilon u(x-y)$ for $\varepsilon = \pm 1$ and $y \in \mathbb{R}$.

## 1.2. The case of the interval

In this section, we consider the case where $\Omega$ is a bounded interval, i.e. $N = 1$ and $R < \infty$. In other words, $\Omega = (-R, R)$. We consider again the equation (1.1.1), but now with the boundary condition

$$u(-R) = u(R) = 0. \tag{1.2.1}$$

The situation is more complex than in the preceding section. Indeed, note first that the condition $g(0) = 0$ is not anymore necessary. For example, in the case $g(u) = 4u - 2$ and $R = \pi$, there is the solution $u(x) = \sin^2 x$. Also, there are necessary conditions involving not only $g$, but relations between $g$ and $R$. For example, let $g(u) = u$. Since in this case all solutions of (1.1.1) have the form $u(x) = a \sin(x+b)$, we see that there is a nontrivial solution of (1.1.1)-(1.2.1) if and only if $R = k\pi/2$ for some positive integer $k$. Moreover, this example shows that, as opposed to the case $R = \infty$, there is not uniqueness of positive (or negative) solutions up to translations.

We give a necessary and sufficient condition on $g$ for the existence of nontrivial solutions of (1.1.1)-(1.2.1). Moreover, we characterize all solutions. The characterization, however, is not as simple as in the case $R = \infty$. In the case of odd nonlinearities, the situation is relatively simple, and we show that all solutions are derived from positive solutions on smaller intervals by reflexion.

We recall some simple properties of the equation (1.1.1) which follow from Remark 1.1.1.

Remark 1.2.1. The following properties hold.

(i) Suppose that $u$ satisfies (1.1.1) on some interval $(a, b)$, that $u(a) = u(b) = 0$ and that $u > 0$ on $(a, b)$. Then $u$ is symmetric with respect to $(a+b)/2$, i.e. $u(x) \equiv u(a+b-x)$, and $u'(x) > 0$ for all $a < x < (a+b)/2$. Similarly, if $u < 0$ on $(a, b)$, then $u$ is symmetric with respect to $(a+b)/2$ and $u'(x) < 0$ for all $a < x < (a+b)/2$. Indeed, suppose that $u'(x_0) = 0$ for some $x_0 \in (a, b)$. Then $u$ is symmetric about $x_0$, by Remark 1.1.1 (iv). If $x_0 < (a+b)/2$, we obtain in particular $u(2x_0 - a) = u(a) = 0$, which is absurd since $u > 0$ on $(a, b)$ and $2x_0 - a \in (a, b)$. We obtain as well a contradiction if $x_0 > (a+b)/2$. Therefore, $(a+b)/2$ is the only zero of $u'$ on $(a, b)$ and $u$ is symmetric with respect to $(a+b)/2$. Since $u > 0$ on $(a, b)$, we must then have $u'(x) > 0$ for all $a < x < (a+b)/2$.

(ii) Suppose again that $u$ satisfies (1.1.1) on some interval $(a, b)$, that $u(a) = u(b) = 0$ and that $u > 0$ on $(a, b)$. Then $g((a+b)/2) > 0$. If instead $u < 0$ on $(a, b)$, then $g((a+b)/2) < 0$. Indeed, it follows from (i) that $u$ achieves its maximum at $(a+b)/2$. In particular, $0 \le u''((a+b)/2) = -g(u((a+b)/2))$. Now, if $g(u((a+b)/2)) = 0$, then $u \equiv u((a+b)/2)$ by uniqueness, which is absurd.

Remark 1.2.2. In view of Remarks 1.1.1 and 1.2.1, we see that any nontrivial solution $u$ of (1.1.1)-(1.2.1) must have a specific form. More precisely, we can make the following observations.

(i) $u$ can be positive or negative, in which case $u$ is even and $|u(x)|$ is decreasing for $x \in (0, R)$ (by Remark 1.2.1 (i)).

(ii) If $u$ is neither positive nor negative, then $u'$ vanishes at least twice in $\Omega$, so that $u$ is the restriction to $\Omega$ of a periodic solution in $\mathbb{R}$ (by Remark 1.1.1).

(iii) Suppose $u$ is neither positive nor negative and let $\tau > 0$ be the minimal period of $u$. Set $w(x) = u(-R + x)$, so that $w(0) = w(\tau) = 0$. Two possibilities may occur.

a) Either $w > 0$ (respectively $w < 0$) on $(0, \tau)$ (and thus $w'(0) = w'(\tau) = 0$ because $u$ is $C^1$). In this case, we clearly have $R = k\tau$ for some integer $k \geq 1$, and so $u$ is obtained by periodicity from a positive (respectively negative) solution ($u$ itself) on the smaller interval $(-R, -R + \tau)$.

b) Else, $w$ vanishes in $(0, \tau)$, and then there exists $\sigma \in (0, \tau)$ such that $w > 0$ (respectively $w < 0$) on $(0, \sigma)$, $w$ is symmetric about $\sigma/2$, $w < 0$ (respectively $w > 0$) on $(\sigma, \tau)$ and $w$ is symmetric about $(\tau + \sigma)/2$. In this case, $u$ is obtained from a positive solution and a negative solution on smaller intervals ($u$ on $(-R, -R + \sigma)$ and $u$ on $(-R + \sigma, -R + \tau)$). The derivatives of these solutions must agree at the endpoints (because $u$ is $C^1$) and $2R = m\sigma + n(\tau - \sigma)$, where $m$ and $n$ are positive integers such that $n = m$ or $n = m + 1$ or $n = m - 1$. To verify this, we need only show that $w$ takes both positive and negative values in $(0, \tau)$ and that $w$ vanishes only once (the other conclusions then follow easily). We first show that $w$ takes values of both signs. Indeed, if for example $w \geq 0$ on $(0, \tau)$, then $w$ vanishes at some $\tau_1 \in (0, \tau)$ and $w'(0) = w'(\tau_1) = w'(\tau) = 0$. Then $w$ is periodic of period $2\tau_1$ and of period $2(\tau - \tau_1)$ by Remark 1.1.1 (v). Since $\tau$ is the minimal period of $w$, we must have $\tau_1 = \tau/2$. Therefore, $w'$ must vanish at some $\tau_2 \in (0, \tau_1)$, and so $w$ has the period $2\tau_2 < \tau$, which is absurd. Finally, suppose $w$ vanishes twice in $(0, \tau)$. This implies that $w'$ has three zeroes $\tau_1 < \tau_2 < \tau_3$ in $(0, \tau)$. By Remark 1.1.1 (v), $w$ is periodic with the periods $2(\tau_2 - \tau_1)$ and $2(\tau_3 - \tau_2)$. We must then have $2(\tau_2 - \tau_1) \geq \tau$ and $2(\tau_3 - \tau_2) \geq \tau$. It follows that $\tau_3 - \tau_1 \geq \tau$, which is absurd.

(iv) Assume $g$ is odd. In particular, there is the trivial solution $u \equiv 0$. Suppose $u$ is neither positive nor negative, $u \not\equiv 0$ and let $\tau > 0$ be the minimal period of $u$. Then it follows from (iii) above that $u(\tau - x) = -u(x)$ for all $x \in [0, \tau]$. Indeed, the first possibility of (iii) cannot occur since if $u(0) = u'(0) = 0$, then $u \equiv 0$ by uniqueness (because $g(0) = 0$). Therefore, the second possibility occurs, but by oddness of $g$ and uniqueness, we must have $\sigma = \tau/2$, and $u(\tau - x) = -u(x)$ for all $x \in [0, \tau]$. In other words, $u$ is obtained from a positive solution on $(-R, -R + \sigma)$, with $\sigma = R/2m$ for some positive integer $m$, which is extended to $(-R, R)$ by successive reflexions.

It follows from the above Remark 1.2.2 that the study of the general nontrivial solution of (1.1.1)-(1.2.1) reduces to the study of positive and negative solutions (for possibly different values of $R$). We now give a necessary and sufficient condition for the existence of such solutions.

Theorem 1.2.3. *There exists a solution $u > 0$ of* (1.1.1)-(1.2.1) *if and only if there exists $u_0 > 0$ such that*

(i) $g(u_0) > 0$;

(ii) $G(u) < G(u_0)$ *for all* $0 < u < u_0$;

(iii) *either* $G(u_0) > 0$ *or else* $G(u_0) = 0$ *and* $g(0) < 0$;

(iv) $\displaystyle\int_0^{u_0} \frac{ds}{\sqrt{2}\sqrt{G(u_0) - G(s)}} = R.$

*In this case, $u > 0$ defined by*

$$\int_{u(x)}^{u_0} \frac{ds}{\sqrt{2}\sqrt{G(u_0) - G(s)}} = |x|, \tag{1.2.2}$$

*for all $x \in \Omega$, satisfies (1.1.1)-(1.2.1). Moreover, any positive solution has the form (1.2.2) for some $u_0 > 0$ satisfying (i)–(ii).*

*Similarly, there exists a solution $u < 0$ of (1.1.1)-(1.2.1) if and only if there exists $v_0 < 0$ such that $g(v_0) < 0$, $G(v_0) < G(v)$ for all $v_0 < v < 0$, $g(0) > 0$ if $G(v_0) = 0$, and*

$$\int_{v_0}^{0} \frac{ds}{\sqrt{2}\sqrt{G(s) - G(v_0)}} = R.$$

*In this case, $u < 0$ defined by*

$$\int_{v_0}^{u(x)} \frac{ds}{\sqrt{2}\sqrt{G(s) - G(v_0)}} = |x|, \tag{1.2.3}$$

*for all $x \in \Omega$, satisfies (1.1.1)-(1.2.1). Moreover, any negative solution has the form (1.2.3) for some $v_0 < 0$ as above.*

Proof. We consider only the case of positive solutions, the other case being similar. We proceed in two steps.

Step 1. The conditions (i)–(iv) are necessary. Let $u_0 = u(0)$. (i) follows from Remark 1.2.1 (ii). Since $u'(0) = 0$ by Remark 1.2.1 (i), it follows from (1.1.3) that

$$\frac{1}{2}u'(x)^2 + G(u(x)) = G(u_0), \tag{1.2.4}$$

for all $x \in (a, b)$. Since $u'(x) \neq 0$ for all $x \in (-R, R)$, $x \neq 0$ (again by Remark 1.2.1 (i)), (1.2.4) implies (ii). It follows from (1.2.4) that $G(u_0) = u'(R)^2/2 \geq 0$. Suppose now $G(u_0) = 0$. If $g(0) > 0$, then (ii) cannot hold, and if $g(0) = 0$, then $u$ cannot vanish (by Theorem 1.1.3). Therefore, we must have $g(0) < 0$, which proves (iii). Finally, it follows from (1.2.4) that

$$u'(x) = -\sqrt{2}\sqrt{G(u_0) - G(u(x))},$$

on $(0, R)$. Therefore, $\frac{d}{dx}F(u(x)) = 1$, where

$$F(y) = \int_{y}^{u_0} \frac{ds}{\sqrt{2}\sqrt{G(u_0) - G(s)}};$$

and so $F(u(x)) = x$, for $x \in (0, R)$. (1.2.2) follows for $x \in (0, R)$. The case $x \in (-R, 0)$ follows by symmetry. Letting now $x = R$ in (1.2.2), we obtain (iv).

Step 2. Conclusion. Suppose (i)–(iv), and let $u$ be defined by (1.2.2). It is easy to verify by a direct calculation that $u$ satisfies (1.1.1) in $\Omega$, and it follows from (iv) that $u(\pm R) = 0$. Finally, the fact that any solution has the form (1.2.2) for some $u_0 > 0$ satisfying (i)–(iv) follows from Step 1. □

Remark 1.2.4. Note that in general there is *not* uniqueness of positive (or negative) solutions. For example, if $R = \pi/2$ and $g(u) = u$, then $u(x) = a\cos x$ is a positive solution for any $a > 0$. In general, any $u_0 > 0$ satisfying (i)–(iv) gives rise to a solution given by (1.2.2). Since $u(0) = u_0$, two distinct values of $u_0$ give rise to two distinct solutions. For some nonlinearities, however, there exists at most one $u_0 > 0$ satisfying (i)–(iv) (see Remarks 1.2.5 and 1.2.6 below).

We now apply the above results to some model cases.

Remark 1.2.5. Consider $g(u) = a + bu$, $a, b \in \mathbb{R}$.

(i) If $b = 0$, then there exists a unique solution $u$ of (1.1.1)-(1.2.1), which is given by $u(x) = a(R^2 - x^2)/2$. This solution has the sign of $a$ and is nontrivial iff $a \neq 0$.

(ii) If $a = 0$ and $b > 0$, then there is a nontrivial solution of (1.1.1)-(1.2.1) if and only if $2\sqrt{b}R = k\pi$ for some positive integer $k$. In this case, any nontrivial solution $u$ of (1.1.1)-(1.2.1) is given by $u(x) = c\sin(\sqrt{b}(x + R))$ for some $c \in \mathbb{R}$, $c \neq 0$. In particular, the set of solutions is a one parameter family.

(iii) If $a = 0$ and $b \le 0$, then the only solution of (1.1.1)-(1.2.1) is $u \equiv 0$.

(iv) If $a \neq 0$ and $b > 0$, then several cases must be considered. If $\sqrt{b}R = (\pi/2)+k\pi$ for some nonnegative integer $k$, then there is no solution of (1.1.1)-(1.2.1). If $\sqrt{b}R = k\pi$ for some positive integer $k$, then there is a nontrivial solution of (1.1.1)-(1.2.1), and all solutions have the form

$$u(x) = \frac{a}{b}\Big(\frac{\cos(\sqrt{b}x)}{\cos(\sqrt{b}R)} - 1\Big) + c\sin(\sqrt{b}x),$$

for some $c \in \mathbb{R}$. In particular, the set of solutions is a one parameter family. If $c = 0$, then $u$ has constant sign and $u'(-R) = u'(R) = 0$. (If in addition $k$ is even, then also $u(0) = u'(0) = 0$.) If $c \neq 0$, then $u$ takes both positive and negative values. If $\sqrt{b}R \neq (\pi/2) + k\pi$ and $\sqrt{b}R \neq k\pi$ for all nonnegative integers $k$, then there is a unique solution of (1.1.1)-(1.2.1) given by the above formula with $c = 0$. Note that this solution has constant sign if $\sqrt{b}R \le \pi$ and changes sign otherwise.

(v) If $a \neq 0$ and $b < 0$, then there is a unique solution of (1.1.1)-(1.2.1) given by

$$u(x) = \frac{a}{b}\Big(1 - \frac{\cosh(\sqrt{-b}x)}{\cosh(\sqrt{-b}R)}\Big).$$

Note that in particular $u$ has constant sign (the sign of $a$) in $\Omega$.

Remark 1.2.6. Consider $g(u) = au + b|u|^{p-1}u$, with $a, b \in \mathbb{R}$, $b \neq 0$ and $p > 1$. Note that in this case, there is always the trivial solution $u \equiv 0$. Note also that $g$ is odd, so that by Remark 1.2.2 (iv) and Theorem 1.2.3, there is a solution of (1.1.1)-(1.2.1) every time there exists $u_0 > 0$ and a positive integer $m$ such that properties (i), (ii) and (iv) of Theorem 1.2.3 are satisfied and such that

$$\int_0^{u_0} \frac{ds}{\sqrt{2}\sqrt{G(u_0) - G(s)}} = \frac{r}{2m}. \tag{1.2.5}$$

Here, $G$ is given by $G(u) = \dfrac{a}{2}u^2 + \dfrac{b}{p+1}|u|^{p+1}$.

(i) If $a \le 0$ and $b < 0$, then there is no $u_0 > 0$ such that $g(u_0) > 0$. In particular, there is no nontrivial solution of (1.1.1)-(1.2.1).

(ii) If $a \ge 0$ and $b > 0$, then $g > 0$ and $G$ is increasing on $[0, \infty)$. Therefore, there is a pair $\pm u$ of nontrivial solutions of (1.1.1)-(1.2.1) every time there is $u_0 > 0$ and an integer $m \ge 1$ such that property (1.2.5) is satisfied. We have

$$\int_0^{u_0} \frac{ds}{\sqrt{2}\sqrt{G(u_0) - G(s)}} = \int_0^1 \frac{dt}{\sqrt{2}\sqrt{\frac{a}{2}(1-t^2) + \frac{b}{p+1}u_0^{p-1}(1-t^{p+1})}} := \phi(u_0). \tag{1.2.6}$$

It is clear that $\phi : [0, \infty) \to (0, \infty)$ is decreasing, that $\phi(\infty) = 0$ and that

$$\phi(0) = \int_0^1 \frac{dt}{\sqrt{2}\sqrt{\frac{a}{2}(1-t^2)}} = \frac{\pi}{2\sqrt{a}} \quad (+\infty \text{ if } a = 0),$$

by using the change of variable $t = \sin\theta$. Therefore, given any integer $m > 2\sqrt{a}R/\pi$, there exists a unique $u_0(k)$ such that (1.2.5) is satisfied. In particular, the set of nontrivial solutions of (1.1.1)-(1.2.1) is a pair of sequences $\pm(u^n)_{n\geq 0}$. We see that there exists a positive solution (which corresponds to $m = 1$) iff $2\sqrt{a}R < \pi$.

(iii) If $a > 0$ and $b < 0$, then both $g$ and $G$ are increasing on $(0, u_*)$ with $u_* = (-a/b)^{\frac{1}{p-1}}$. On $(u_*, \infty)$, $g$ is negative and $G$ is decreasing. Therefore, the assumptions (i)–(iii) of Theorem 1.2.3 are satisfied iff $u_0 \in (0, u^*)$. Therefore, there is a pair $\pm u$ of nontrivial solutions of (1.1.1)-(1.2.1) every time there is $u_0 \in (0, u_*)$ and an integer $m \geq 1$ such that property (1.2.5) is satisfied. Note that for $u_0 \in (0, u_*)$, formula (1.2.6) holds, but since $b < 0$, $\phi$ is now increasing on $(0, u_*)$, $\phi(0) = \pi/2\sqrt{a}$ and $\phi(u_*) = +\infty$. Therefore, there exists nontrivial solutions iff $2\sqrt{a}R > \pi$, and in this case, there exists a unique positive solution. Moreover, still assuming $2\sqrt{a}R > \pi$, the set of nontrivial solutions of (1.1.1)-(1.2.1) consists of $\ell$ pairs of solutions, where $\ell$ is the integer part of $2\sqrt{a}R/\pi$. Every pair of solution corresponds to some integer $m \in \{1, \dots, \ell\}$ and $u_0 \in (0, u_*)$ defined by $\phi(u_0) = R/2m$.

(iv) If $a < 0$ and $b > 0$, then assumptions (i)–(iii) of Theorem 1.2.3 are satisfied iff $u_0 > u^*$ with $u^* = (-a(p+1)/2b)^{\frac{1}{p-1}}$. Therefore, there is a pair $\pm u$ of nontrivial solutions of (1.1.1)-(1.2.1) every time there is $u_0 > u^*$ and an integer $m \geq 1$ such that property (1.2.5) is satisfied. Note that for $u_0 > u^*$, formula (1.2.6) holds, and that $\phi$ is decreasing on $(u^*, \infty)$, $\phi(u^*) = +\infty$ and $\phi(\infty) = 0$. Therefore, given any integer $k > 2\sqrt{a}R/\pi$, there exists a unique $u_0(k)$ such that (1.2.5) is satisfied. In particular, the set of nontrivial solutions of (1.1.1)-(1.2.1) is a pair of sequences $\pm(u^n)_{n\geq 0}$. We see that there always exists a positive solution (which corresponds to $m = 1$).

## 1.3. The case of $\mathbb{R}^N$, $N \geq 2$

In this section, we look for radial solutions of the equation

$$\begin{cases} -\triangle u = g(u) & \text{in} \quad \mathbb{R}^N, \\ u(x) \longrightarrow_{|x|\to\infty} 0. \end{cases}$$

As observed before, the equation for $u(r) = u(|x|)$ becomes the ODE

$$u'' + \frac{N-1}{r}u' + g(u) = 0, \quad r > 0,$$

with the boundary condition $u(r) \underset{r\to\infty}{\longrightarrow} 0$. For simplicity, we consider the model case

$$g(u) = -\lambda u + \mu |u|^{p-1}u.$$

(One can handle more general nonlinearities by the method we will use, see McLeod, Troy and Weissler [38].) Therefore, we look for solutions of the ODE

$$u'' + \frac{N-1}{r}u' - \lambda u + \mu |u|^{p-1}u = 0, \tag{1.3.1}$$

for $r > 0$ such that

$$u(r) \underset{r\to\infty}{\longrightarrow} 0. \tag{1.3.2}$$

Due to the presence of the nonautonomous term $(N-1)u'/r$ in the equation (1.3.1), this problem turns out to be considerably more difficult than in the one-dimensional case. On the other hand, it has a richer structure, in the sense that there are “more” solutions.

We observe that, given $u_0 > 0$, there exists a unique, maximal solution $u \in C^2([0, R_{\mathrm{m}}))$ of (1.3.1) with the initial conditions $u(0) = u_0$ and $u'(0) = 0$, with the blow up alternative that either $R_{\rm m} = \infty$ or else $|u(r)| + |u'(r)| \to \infty$ as $r \uparrow R_{\rm m}$. To see this, we write the equation in the form

$$(r^{N-1}u'(r))' = \lambda r^{N-1}(u(r) - \mu |u(r)|^{p-1}u(r)), \tag{1.3.3}$$

thus, with the initial conditions,

$$u(r) = u_0 + \int_0^r s^{-(N-1)} \int_0^s \sigma^{N-1}(u(\sigma) - \mu |u(\sigma)|^{p-1}u(\sigma))\, d\sigma\, ds. \tag{1.3.4}$$

This last equation is solved by the usual fixed point method. For $r > 0$, the equation is not anymore singular, so that the solution can be extended by the usual method to a maximal solution which satisfies the blow up alternative.

The nonautonomous term in the equation introduces some dissipation. To see this, let $u$ be a solution on some interval $(a, b)$, with $0 < a < b < \infty$, and set

$$E(u, r) = \frac{1}{2}u'(r)^2 - \frac{\lambda}{2}u(r)^2 + \frac{\mu}{p+1}|u(r)|^{p+1}. \tag{1.3.5}$$

Multiplying the equation by $u'(r)$, we obtain

$$\frac{dE}{dr} = -\frac{N-1}{r}u'(r)^2, \tag{1.3.6}$$

so that $E(u, r)$ is a decreasing quantity.

Note that if $\mu > 0$, there is a constant $C$ depending only on $p, \mu, \lambda$ such that

$$E(u, r) \geq \frac{1}{2}(u'(r)^2 + u(r)^2) - C.$$

In particular, all the solutions of (1.3.1) exist for all $r > 0$ and stay bounded as $r \to \infty$.

The first result of this section is the following.

THEOREM 1.3.1. *Assume $\lambda, \mu > 0$ and $(N-2)p < N+2$. There exists $x_0 > 0$ such that the solution $u$ of (1.3.1) with the initial conditions $u(0) = x_0$ and $u'(0) = 0$ is defined for all $r > 0$, is positive and decreasing. Moreover, there exists $C$ such that*

$$u(r)^2 + u'(r)^2 \leq Ce^{-2\sqrt{\lambda}r}, \tag{1.3.7}$$

*for all $r > 0$.*

When $N = 1$ (see Section 1.1), there is only one radial solution such that $u(0) > 0$ and $u(r) \to 0$ as $r \to \infty$. When $N \geq 2$, there are infinitely many such solutions. More precisely, there is at least one such solution with any prescribed number of nodes, as shows the following result.

THEOREM 1.3.2. *Assume $\lambda, \mu > 0$ and $(N-2)p < N+2$. There exists an increasing sequence $(x_n)_{n\geq 0}$ of positive numbers such that the solution $u_n$ of (1.3.1) with the initial conditions $u_n(0) = x_n$ and $u_n'(0) = 0$ is defined for all $r > 0$, has exactly $n$ nodes, and satisfies for some constant $C$ the estimate (1.3.7).*

We use the method of McLeod, Troy and Weissler [38] to prove the above results. The proof is rather long and relies on some preliminary informations on the equations, which we collect below.

PROPOSITION 1.3.3. *If $u$ is the solution of*

$$\begin{cases} u'' + \frac{N-1}{r}u' + |u|^{p-1}u = 0, \\ u(0) = 1, \quad u'(0) = 0, \end{cases} \tag{1.3.8}$$

*then the following properties hold.*

(i) *If* $N \geq 3$ *and* $(N-2)p \geq N+2$, *then* $u(r) > 0$ *and* $u'(r) < 0$ *for all* $r > 0$. *Moreover,* $u(r) \to 0$ *as* $r \to \infty$.

(ii) *If* $(N-2)p < N+2$, *then* $u$ *oscillates indefinitely. More precisely, for any* $r_0 \geq 0$ *such that* $u(r_0) \neq 0$, *there exists* $r_1 > r_0$ *such that* $u(r_0)u(r_1) < 0$.

PROOF. We note that $u''(0) < 0$, so that $u'(r) < 0$ for $r > 0$ and small. Now, if $u'$ would vanish while $u$ remains positive, we would obtain $u'' < 0$ from the equation, which is absurd. So $u' < 0$ while $u$ remains positive. Next, we deduce from the equation that

$$\Big(\frac{u'^2}{2} + \frac{|u|^{p+1}}{p+1}\Big)' = -\frac{N-1}{r}u'^2, \tag{1.3.9}$$

$$(r^{N-1}uu')' + r^{N-1}|u|^{p+1} = r^{N-1}u'^2, \tag{1.3.10}$$

and

$$\Big(\frac{r^N}{2}u'^2 + \frac{r^N}{p+1}|u|^{p+1}\Big)' + \frac{N-2}{2}r^{N-1}u'^2 = \frac{N}{p+1}r^{N-1}|u|^{p+1}. \tag{1.3.11}$$

We first prove property (i). Assume by contradiction that $u$ has a first zero $r_0$. By uniqueness, we have $u'(r_0) \neq 0$. Integrating (1.3.10) and (1.3.11) on $(0, r_0)$, we obtain

$$\int_0^{r_0} r^{N-1}u^{p+1} = \int_0^{r_0} r^{N-1}u'^2,$$

and

$$\frac{r_0^N}{2}u'(r_0)^2 + \frac{N-2}{2}\int_0^{r_0} r^{N-1}u'^2 = \frac{N}{p+1}\int_0^{r_0} r^{N-1}|u|^{p+1};$$

and so,

$$0 < \frac{r_0^N}{2}u'(r_0)^2 = \Big(\frac{N}{p+1} - \frac{N-2}{2}\Big)\int_0^{r_0} r^{N-1}u'^2 \leq 0,$$

which is absurd. This shows that $u(r) > 0$ (hence $u'(r) < 0$) for all $r > 0$. In particular, $u(r)$ decreases to a limit $\ell \geq 0$ as $r \to \infty$. Since $u'(r)$ is bounded by (1.3.9), we deduce from the equation that $u''(r) \to -\ell^p$, which implies that $\ell = 0$. This proves property (i)

We now prove property (ii), and we first show that $u$ must have a first zero. Indeed, suppose by contradiction that $u(r) > 0$ for all $r > 0$. It follows that $u'(r) < 0$ for all $r > 0$. Thus $u$ has a limit $\ell \geq 0$ as $r \to \infty$. Note that by (1.3.6), $u'$ is bounded, so that by the equation $u''(r) \to -\ell^p$ as $r \to \infty$, which implies that $\ell = 0$. We write equation (1.3.8) in the form

$$r^{N-1}u'(r) = -\int_0^r s^{N-1}u^p(s); \tag{1.3.12}$$

and so

$$-r^{N-1}u'(r) = \int_0^r s^{N-1}u^p \geq u(r)^p \int_0^r s^{N-1} = \frac{r^N}{N}u(r)^p.$$

Therefore,

$$\Big(\frac{1}{(p-1)u(r)^{p-1}} - \frac{r^2}{2N}\Big)' \geq 0,$$

which implies that

$$u(r) \leq Cr^{-\frac{2}{p-1}}. \tag{1.3.13}$$

By the assumption on $p$, this implies that

$$\int_0^\infty r^{N-1}u(r)^{p+1} < \infty. \tag{1.3.14}$$

If $N = 2$, then (1.3.12)-(1.3.13) show that $ru'(r)$ converges to a negative limit as $r \to \infty$, which is absurd. We now suppose $N \ge 3$ and we integrate (1.3.11) on $(0, r)$:

$$\frac{r^N}{2}u'(r)^2 + \frac{r^N}{p+1}u(r)^{p+1} + \frac{N-2}{2}\int_0^r s^{N-1}u'^2 = \frac{N}{p+1}\int_0^r s^{N-1}u^{p+1}. \quad (1.3.15)$$

Letting $r \to \infty$ and applying (1.3.14), we deduce that

$$\int_0^\infty r^{N-1}u'(r)^2 < \infty. \quad (1.3.16)$$

It follows in particular from (1.3.14) and (1.3.16) that there exist $r_n \to \infty$ such that

$$r_n^N((u'(r_n)^2 + u(r_n)^{p+1}) \to 0.$$

Letting $r = r_n$ in (1.3.15) and applying (1.3.14) and (1.3.16), we deduce by letting $n \to \infty$

$$\frac{N-2}{2}\int_0^\infty s^{N-1}u'^2 = \frac{N}{p+1}\int_0^\infty s^{N-1}u^{p+1}. \quad (1.3.17)$$

Finally, we integrate (1.3.10) on $(0, r)$:

$$r^{N-1}u(r)u'(r) + \int_0^r s^{N-1}u^{p+1} = \int_0^r s^{N-1}u'^2. \quad (1.3.18)$$

We observe that $u(r_n) \le cr_n^{-\frac{N}{p+1}}$ and that $|u'(r_n)| \le cr_n^{-\frac{N}{2}}$. By the assumption on $p$, this implies that $r_n^{N-1}u(r_n)u'(r_n) \to 0$. Letting $r = r_n$ in (1.3.18) and letting $n \to \infty$, we obtain

$$\int_0^\infty s^{N-1}u^{p+1} = \int_0^\infty s^{N-1}u'^2.$$

Multiplying the above identity by $N/(p+1)$ and making the difference with (1.3.17), we obtain

$$0 = \Bigl(\frac{N}{p+1} - \frac{N-2}{2}\Bigr)\int_0^\infty r^{N-1}u'^2 > 0,$$

which is absurd.

In fact, with the previous argument, one shows as well that if $r \ge 0$ is such that $u(r) \ne 0$ and $u'(r) = 0$, then there exists $\rho > r$ such that $u(\rho) = 0$.

To conclude, we need only show that if $\rho > 0$ is such that $u(\rho) = 0$, then there exists $r > \rho$ such that $u(r) \ne 0$ and $u'(r) = 0$. To see this, note that $u'(\rho) \ne 0$ (for otherwise $u \equiv 0$ by uniqueness), and suppose for example that $u'(\rho) > 0$. If $u'(r) > 0$ for all $r \ge \rho$, then (since $u$ is bounded) $u$ converges to some positive limit $\ell$ as $r \to \infty$; and so, by the equation, $u''(r) \to -\ell^p$ as $r \to \infty$, which is absurd. This completes the proof. □

REMARK 1.3.4. Here are some comments on Proposition 1.3.3 and its proof.

(i) Property (ii) does not hold for singular solutions of (1.3.8). Indeed, for $p > N/(N-2)$, there is the (singular) solution

$$u(r) = \Bigl(\frac{(N-2)p - N}{2}\Bigr)^{\frac{1}{p-1}}\Bigl(\frac{2}{(p-1)r}\Bigr)^{\frac{2}{p-1}}, \quad (1.3.19)$$

which is positive for all $r > 0$.

(ii) The argument at the beginning of the proof of property (ii) shows that any positive solution $u$ of (1.3.8) on $[R, \infty)$ $(R \ge 0)$ satisfies the estimate (1.3.13) for $r$ large. This holds for any value of $p$. The explicit solutions (1.3.19) show that this estimate cannot be improved in general.

(iii) Let $p > 1$, $N \geq 3$ and let $u$ be a positive solution of (1.3.8) on $(R, \infty)$ for some $R > 0$. If $u(r) \to 0$ as $r \to \infty$, then there exists $c > 0$ such that

$$u(r) \geq \frac{c}{r^{N-2}}, \tag{1.3.20}$$

for all $r \geq R$. Indeed, $(r^{N-1}u')' = -r^{N-1}u^p \leq 0$, so that

$$u'(r) \leq R^{N-1}u'(R)r^{-(N-1)}.$$

Integrating on $(r, \infty)$, we obtain $(N-2)r^{N-2}u(r) \geq -R^{N-1}u'(R)$. Since $u > 0$ and $u(r) \to 0$ as $r \to \infty$, we may assume without loss of generality that $u'(R) < 0$ and (1.3.20) follows.

Corollary 1.3.5. *Assume $\lambda, \mu > 0$ and $(N-2)p < N+2$. For any $\rho > 0$ and any $n \in \mathbb{N}$, $n \geq 1$, there exists $M_{n,\rho}$ such that if $x_0 > M_{n,\rho}$, then the solution $u$ of (1.3.1) with the initial conditions $u(0) = x_0$ and $u'(0) = 0$ has at least $n$ zeroes on $(0, \rho)$.*

Proof. Changing $u(r)$ to $(\mu/\lambda)^{\frac{1}{p-1}}u(\lambda^{-\frac{1}{2}}r)$, we are reduced to the equation

$$u'' + \frac{N-1}{r}u' - u + |u|^{p-1}u = 0. \tag{1.3.21}$$

Let now $R > 0$ be such that the solution $v$ of (1.3.8) has $n$ zeroes on $(0, R)$ (see Proposition 1.3.3).

Let $x > 0$ and let $u$ be the solution of (1.3.21) such that $u(0) = x$, $u'(0) = 0$. Set

$$\widetilde{u}(r) = \frac{1}{x}u\Big(\frac{r}{x^{\frac{p-1}{2}}}\Big),$$

so that

$$\begin{cases} \widetilde{u}'' + \frac{N-1}{r}\widetilde{u}' - \frac{1}{x^{p-1}}\widetilde{u} + |\widetilde{u}|^{p-1}\widetilde{u} = 0, \\ \widetilde{u}(0) = 1, \quad \widetilde{u}'(0) = 0. \end{cases}$$

It is not difficult to show that $\widetilde{u} \to v$ in $C^1([0, R])$ as $x \to \infty$. Since $v' \neq 0$ whenever $v = 0$, this implies that for $x$ large enough, say $x \geq x_n$, $\widetilde{u}$ has $n$ zeroes on $(0, R)$. Coming back to $u$, this means that $u$ has $n$ zeroes on $(0, (R/x)^{\frac{p-1}{2}})$. The result follows with for example $M_{n,\rho} = \max\{x_n, (R/\rho)^{\frac{2}{p-1}}\}$. ☐

Lemma 1.3.6. *For every $c > 0$, there exists $\alpha(c) > 0$ with the following property. If $u$ is a solution of (1.3.1) and if $E(u, R) = -c < 0$ and $u(R) > 0$ for some $R \geq 0$ ($E$ is defined by (1.3.5)), then $u(r) \geq \alpha(c)$ for all $r \geq R$.*

Proof. Let $f(x) = \mu|x|^{p+1}/(p+1) - \lambda x^2/2$ for $x \in \mathbb{R}$, and let $-m = \min f < 0$. One verifies easily that for every $c \in (0, m)$ the equation $f(x) = -c$ has two positive solutions $0 < \alpha(c) \leq \beta(c)$, and that if $f(x) \leq -c$, then $x \in [-\beta(c), -\alpha(c)] \cup [\alpha(c), \beta(c)]$. It follows from (1.3.6) that $f(u(r)) \leq -c$ for all $r \geq R$, from which the result follows immediately. ☐

We are now in a position to prove Theorem 1.3.1.

Proof of Theorem 1.3.1. Let

$$A_0 = \{x > 0;\ u > 0 \text{ on } (0, \infty)\},$$

where $u$ is the solution of (1.3.1) with the initial values $u(0) = x$, $u'(0) = 0$.

We claim that $I = (0, (\lambda(p+1)/2\mu)^{\frac{1}{p-1}}) \subset A_0$, so that $A_0 \neq \emptyset$. Indeed, suppose $x \in I$. It follows that $E(u, 0) < 0$; and so, $\inf\limits_{r \geq 0} u(r) > 0$ by Lemma 1.3.6. On the other hand, $A_0 \subset (0, M_{1,1})$ by Corollary 1.3.5. Therefore, we may consider $x_0 = \sup A_0$. We claim that $x_0$ has the desired properties.

Indeed, let $u$ be the solution with initial value $x_0$. We first note that $x_0 \in A_0$. Otherwise, $u$ has a first zero at some $r_0 > 0$. By uniqueness, $u'(r_0) \neq 0$, so that $u$ takes negative values. By continuous dependance, this is the case for solutions with initial values close to $x_0$, which contradicts the property $x_0 \in \overline{A_0}$. On the other hand, we have $x_0 > (\lambda(p+1)/2\mu)^{\frac{1}{p-1}} > (\lambda/\mu)^{\frac{1}{p-1}}$. This implies that $u''(0) < 0$, so that $u'(r) < 0$ for $r > 0$ and small. We claim that $u'(r)$ cannot vanish. Otherwise, for some $r_0 > 0$, $u(r_0) > 0$, $u'(r_0) = 0$ and $u''(r_0) \geq 0$. This implies that $u(r_0) \leq (\lambda/\mu)^{\frac{1}{p-1}}$, which in turn implies $E(u, r_0) < 0$. By continuous dependance, it follows that for $v_0$ close to $x_0$, we have $E(v, r_0) < 0$, which implies that $v_0 \in A_0$ by Lemma 1.3.6. This contradicts again the property $x_0 = \sup A_0$. Thus $u'(r) < 0$ for all $r > 0$. Let

$$m = \inf_{r \geq 0} u(r) = \lim_{r \to \infty} u(r) \geq 0$$

We claim that $m = 0$. Indeed if $m > 0$, we deduce from the equation that (since $u'$ is bounded)

$$u''(r) \underset{r \to \infty}{\longrightarrow} \lambda m - \mu m^p.$$

Thus, either $m = 0$ or else $m = (\lambda/\mu)^{\frac{1}{p-1}}$. In this last case, since $u'(r_n) \to 0$ for some sequence $r_n \to \infty$, we have $\liminf E(u, r) < 0$ as $r \to \infty$, which is again absurd by Lemma 1.3.6. Thus $m = 0$. The exponential decay now follows from the next lemma (see also Proposition 4.4.9 for a more general result). □

LEMMA 1.3.7. *Assume $\lambda, \mu > 0$. If $u$ is a solution of (1.3.1) on $[r_0, \infty)$ such that $u(r) \to 0$ as $r \to \infty$, then there exists a constant $C$ such that*

$$u(r)^2 + u'(r)^2 \leq Ce^{-2\sqrt{\lambda} r},$$

*for $r \geq r_0$.*

PROOF. Let $v(r) = (\mu/\lambda)^{\frac{1}{p-1}} u(\lambda^{-\frac{1}{2}} r)$, so that $v$ is a solution of (1.3.21). Set

$$f(r) = v(r)^2 + v'(r)^2 - 2v(r)v'(r).$$

We see easily that for $r$ large enough $v(r)v'(r) < 0$, so that, by possibly chosing $r_0$ larger,

$$f(r) \geq v(r)^2 + v'(r)^2, \tag{1.3.22}$$

for $r \geq r_0$. An elementary calculation shows that

$$\begin{aligned} f'(r) + 2f(r) &= -\frac{2(N-1)}{r}(v'^2 - vv') + 2|v|^{p-1}(v^2 - vv') \\ &\leq 2|v|^{p-1}(v^2 - vv') \leq 2|v|^{p-1} f. \end{aligned}$$

It follows that

$$\frac{f'(r)}{f(r)} + 2 - 2|v|^{p-1} \leq 0;$$

and so, given $r_0$ sufficiently large,

$$\frac{d}{dr}\Big( \log(f(r)) + 2r - 2\int_{r_0}^{r} |v|^{p-1} \Big) \leq 0.$$

Since $v$ is bounded, we first deduce that $f(r) \leq Ce^{-r}$. Applying the resulting estimate $|v(r)| \leq Ce^{-\frac{r}{2}}$ in the above inequality, we now deduce that $f(r) \leq Ce^{-2r}$. Using (1.3.22), we obtain the desired estimate. □

Finally, for the proof of Theorem 1.3.2, we will use the following lemma.

LEMMA 1.3.8. *Let $n \in \mathbb{N}$, $x > 0$, and let $u$ be the solution of (1.3.1) with the initial conditions $u(0) = x$ and $u'(0) = 0$. Assume that $u$ has exactly $n$ zeroes on $(0, \infty)$ and that $u^2 + u'^2 \to 0$ as $r \to \infty$. There exists $\varepsilon > 0$ such that if $|x - y| \leq \varepsilon$, then the corresponding solution $v$ of (1.3.1) has at most $n + 1$ zeroes on $(0, \infty)$.*

PROOF. Assume for simplicity that $\lambda = \mu = 1$. We first observe that $E(u, r) > 0$ for all $r > 0$ by Lemma 1.3.6. This implies that if $r > 0$ is a zero of $u'$, then $|u(r)|^{p-1} > (p + 1)/2 > 1$, so that $u(r)u''(r) < 0$, by the equation. In particular, if $r_2 > r_1$ are two consecutive zeroes of $u'$, it follows that $u(r_1)u(r_2) < 0$, so that $u$ has a zero in $(r_1, r_2)$. Therefore, since $u$ has a finite number of zeroes, it follows that $u'$ has a finite number of zeroes.

Let $r' \geq 0$ be the largest zero of $u'$ and assume, for example, that $u(r') > 0$. In particular, $u(r') > 1$ and $u$ is decreasing on $[r', \infty)$. Therefore, there exists a unique $r_0 \in (r', \infty)$ such that $u(r_0) = 1$, and we have $u'(r_0) < 0$. By continuous dependance, there exists $\varepsilon > 0$ such that if $|x - y| \leq \varepsilon$, and if $v$ is the solution of (1.3.1) with the initial conditions $v(0) = x$, then the following properties hold.

(i) There exists $\rho_0 \in [r_0 - 1, r_0 + 1]$ such that $v$ has exactly $n$ zeroes on $[0, \rho_0]$
(ii) $v(\rho_0) = 1$ and $v'(\rho_0) < 0$.

Therefore, we need only show that, by choosing $\varepsilon$ possibly smaller, $v$ has at most one zero on $[\rho_0, \infty)$. To see this, we suppose $v$ has a first zero $\rho_1 > \rho_0$, and we show that if $\varepsilon$ is small enough, then $v < 0$ on $(\rho_1, \infty)$. Since $v(\rho_1) = 0$, we must have $v'(\rho_1) < 0$; and so, $v'(r) < 0$ for $r - \rho_1 > 0$ and small. Furthermore, it follows from the equation that $v'$ cannot vanish while $v > -1$. Therefore, there exist $\rho_3 > \rho_3 > \rho_1$ such that $v' < 0$ on $[\rho_1, \rho_3]$ and $v(\rho_2) = -1/4$, $v(\rho_3) = -1/2$. By Lemma 1.3.6, we obtain the desired result if we show that $E(v, \rho_3) < 0$ provided $\varepsilon$ is small enough. To see this, we first observe that, since $u > 0$ on $[r', \infty)$,

$$\forall M > 0, \exists \varepsilon' \in (0, \varepsilon) \text{ such that } \rho_1 > M \text{ if } |x - y| \leq \varepsilon'.$$

Let

$$f(x) = \frac{|x|^{p+1}}{p+1} - \frac{x^2}{2}.$$

It follows from (1.3.6) that

$$\frac{d}{dr}E(v, r) + \frac{2(N-1)}{r}E(v, r) = \frac{2(N-1)}{r}f(v(r));$$

and so,

$$\frac{d}{dr}(r^{2(N-1)}E(v, r)) = 2(N-1)r^{2N-3}f(v(r)).$$

Integrating on $(\rho_0, \rho_3)$, we obtain

$$\rho_3^{2(N-1)}E(v, \rho_3) = \rho_0^{2(N-1)}E(v, \rho_0) + 2(N-1)\int_{\rho_0}^{\rho_3} r^{2N-3}f(v(r))\,dr.$$

Note that (by continuous dependence)

$$\rho_0^{2(N-1)}E(v, \rho_0)^{2(N-1)} \leq C,$$

with $C$ independent of $y \in (x - \varepsilon, x + \varepsilon)$. On the other hand, $f(v(r)) \leq 0$ on $(\rho_0, \rho_3)$ since $-1 \leq v \leq 1$, and there exists $a > 0$ such that $f(\theta) \leq -a$ for $\theta \in (-1/4, -1/2)$. It follows that

$$\begin{aligned}\rho_3^{2(N-1)}E(v, \rho_3) &\leq C - 2(N-1)a\int_{\rho_2}^{\rho_3} r^{2N-3}\,dr \\ &\leq C - 2(N-1)a\rho_2^{2N-3}(\rho_3 - \rho_2).\end{aligned}$$

Since $v'$ is bounded on $(\rho_2, \rho_3)$ independently of $y$ such that $|x-y| \le \varepsilon'$, it follows that $\rho_3 - \rho_2$ is bounded from below. Therefore, we see that $E(v, \rho_3) < 0$ if $\varepsilon$ is small enough, which completes the proof. □

Proof of Theorem 1.3.2. Let

$$A_1 = \{x > x_0;\ u \text{ has exactly one zero on } (0,\infty)\}.$$

By definition of $x_0$ and Lemma 1.3.8, we have $A_1 \neq \emptyset$. In addition, it follows from Corollary 1.3.5 that $A_1$ is bounded. Let

$$x_1 = \sup A_1,$$

and let $u_1$ be the corresponding solution. By using the argument of the proof of Theorem 1.3.1, one shows easily that $u_1$ has the desired properties. Finally, one defines by induction

$$A_{n+1} = \{x > x_n;\ u \text{ has exactly } n+1 \text{ zeroes on } (0,\infty\},$$

and

$$x_{n+1} = \sup A_{n+1},$$

and one show that the corresponding solution $u_n$ has the desired properties. □

Remark 1.3.9. Here are some comments on the cases when the assumptions of Theorems 1.3.1 and 1.3.2 are not satisfied.

(i) If $\lambda, \mu > 0$ and $(N-2)p \ge N+2$, then there does not exist any solution $u \not\equiv 0$, $u \in C^1([0,\infty))$ of (1.3.1)-(1.3.2). Indeed, suppose for simplicity $\lambda = \mu = 1$ and assume by contradiction that there is a solution $u$. Arguing as in the proof of Lemma 1.3.7, one shows easily that $u$ and $u'$ must have exponential decay. Next, arguing as in the proof of Proposition 1.3.3, one shows that

$$\int_0^\infty s^{N-1}|u|^{p+1} = \int_0^\infty s^{N-1}u'^2 + \int_0^\infty s^{N-1}u^2,$$

and

$$\frac{N}{p+1}\int_0^\infty s^{N-1}|u|^{p+1} = \frac{N-2}{2}\int_0^\infty s^{N-1}u'^2 + \frac{N}{2}\int_0^\infty s^{N-1}u^2.$$

It follows that

$$0 < \Big(\frac{N-2}{2} - \frac{N}{p+1}\Big)\int_0^\infty s^{N-1}|u|^{p+1} = -\int_0^\infty s^{N-1}u^2 < 0,$$

which is absurd.

(ii) If $\lambda > 0$ and $\mu < 0$, then there does not exist any solution $u \not\equiv 0$, $u \in C^1([0,\infty))$ of (1.3.1)-(1.3.2). Indeed, suppose for example $\lambda = 1$ and $\mu = -1$ and assume by contradiction that there is a solution $u$. Since $E(u,r)$ is decreasing and $u \to 0$, we see that $u'$ is bounded. It then follows the equation that $u'' \to 0$ as $r \to \infty$; and so, $u' \to 0$ (see Step 1 of the proof of Theorem 1.1.3). Therefore, $E(u,r) \to 0$ as $r \to \infty$, and since $E(u,r)$ is nonincreasing, we must have in particular $E(u,0) \ge 0$. This is absurd, since $E(u,0) = -u(0)^2/2 - u(0)^{p+1}/(p+1) < 0$.

(iii) If $\lambda = 0$ and $\mu < 0$, then there does not exist any solution $u \not\equiv 0$, $u \in C^1([0,\infty))$ of (1.3.1)-(1.3.2). This follows from the argument of (ii) above.

(iv) If $\lambda = 0$, $\mu > 0$ and $(N-2)p = N+2$, then for any $x > 0$ the solution $u$ of (1.3.1) such that $u(0) = x$ is given by

$$u(r) = x\Big(1 + \frac{\mu x^{\frac{4}{N-2}}}{N(N-2)}r^2\Big)^{-\frac{N-2}{2}}.$$

In particular, $u(r) \approx r^{-(N-2)}$ as $r \to \infty$. Note that $u \in L^{p+1}(\mathbb{R}^N)$. In addition, $u \in H^1(\mathbb{R}^N)$ if and only if $N \ge 5$.

(v) If $\lambda = 0$, $\mu > 0$ and $(N-2)p > N+2$, then for any $x > 0$ the solution $u$ of (1.3.1) such that $u(0) = x$ satisfies (1.3.2). (This follows from Proposition 1.3.3.) However, $u$ has a slow decay as $r \to \infty$ in the sense that $u \notin L^{p+1}(\mathbb{R}^N)$. Indeed, if $u$ were in $L^{p+1}(\mathbb{R}^N)$, then arguing as in the proof of Proposition 1.3.3 (starting with (1.3.14)) we would get to a contradiction.

(vi) If $\lambda = 0$, $\mu > 0$ and $(N-2)p < N+2$, then for any $x > 0$ the solution $u$ of (1.3.1) such that $u(0) = x$ satisfies (1.3.2). However, $u$ has a slow decay as $r \to \infty$ in the sense that $u \notin L^{p+1}(\mathbb{R}^N)$. This last property follows from the argument of (v) above. The property $u(r) \to 0$ as $r \to \infty$ is more delicate, and one can proceed as follows. We show by contradiction that $E(u,r) \to 0$ as $r \to \infty$. Otherwise, since $E(u,r)$ is nonincreasing, $E(u,r) \downarrow \ell > 0$ as $r \to \infty$. Let $0 < r_1 < r_2 \leq \dots$ be the zeroes of $u$ (see Proposition 1.3.3). We deduce that $u'(r_n)^2 \to 2\ell$ as $n \to \infty$. Consider the solution $\omega$ of the equation $\omega'' + \mu|\omega|^{p-1}\omega = 0$ with the initial values $\omega(0) = 0$, $\omega'(0) = \sqrt{2\ell}$. $\omega$ is anti-periodic with minimal period $2\tau$ for some $\tau > 0$. By a continuous dependence argument, one shows that $r_{n+1} - r_n \to \tau$ as $n \to \infty$ and that $|u(r_n + \cdot) - \omega(\cdot) \operatorname{sign} u'(r_n)| \to 0$ in $C^1([0,\tau])$. This implies that $r_n \leq 2n\tau$ for $n$ large and that

$$\int_{r_n}^{r_{n+1}} u'(r)^2\, dr \geq \frac{1}{2} \int_0^\tau \omega'(r)^2\, dr \geq \delta > 0,$$

for some $\delta > 0$ and $n$ large. It follows that

$$\int_{r_n}^{r_{n+1}} \frac{u'(r)^2}{r} dr \geq \frac{\delta}{r_{n+1}} \geq \frac{\delta}{2\tau(n+1)}.$$

We deduce that

$$\int_0^\infty \frac{u'(r)^2}{r} = +\infty,$$

which yields a contradiction (see (1.3.6)).

(vii) If $\lambda < 0$, then there does not exist any solution $u$ of (1.3.1) with $u \in L^2(\mathbb{R}^N)$. This result is delicate. It is proved in Kato [27] in a more general setting (see also Agmon [2]). We follow here the less general, but much simpler argument of Lopes [34]. We consider the case $\mu < 0$, which is slightly more delicate, and we assume for example $\lambda = \mu = -1$. Setting $\varphi(r) = r^{\frac{N-1}{2}} u(r)$, we see that

$$\varphi'' + \varphi = \frac{(N-1)(N-3)}{4r^2}\varphi + r^{-\frac{(N-1)(p-1)}{2}} |\varphi|^{p-1}\varphi.$$

Setting

$$\begin{aligned} H(r) &= \frac{1}{2}\varphi'^2 + \frac{1}{2}\varphi^2 - \frac{(N-1)(N-3)}{8r^2}\varphi^2 - \frac{1}{p+1} r^{-\frac{(N-1)(p-1)}{2}} |\varphi|^{p+1} \\ &= \frac{1}{2}\varphi'^2 + \frac{1}{2}\varphi^2 \Big[ 1 - \frac{(N-1)(N-3)}{8r^2} - \frac{|u|^{p-1}}{p+1} \Big], \end{aligned}$$

we deduce that

$$\begin{aligned} H'(r) &= \frac{(N-1)(N-3)}{4r^3}\varphi^2 + \frac{(N-1)(p-1)}{2(p+1)} r^{-\frac{(N-1)(p-1)}{2} - 1} |\varphi|^{p+1} \\ &= \Big( \frac{(N-1)(N-3)}{4r^3} + \frac{(N-1)(p-1)}{2(p+1)r} |u|^{p-1} \Big) \varphi^2. \end{aligned}$$

Since $u(r) \to 0$ as $r \to \infty$, we deduce from the above identities that for any $\varepsilon > 0$, we have

$$H'(r) \leq \frac{\varepsilon}{r} H(r),$$

for $r$ large enough, which implies that $H(r) \le C_\varepsilon r^\varepsilon$.In particular, $|u(r)| \le Cr^{-\frac{N-1-\varepsilon}{2}}$. Therefore,

$$H'(r) \le C(r^{-3} + r^{-1-\frac{(N-1-\varepsilon)(p-1)}{2}})H(r),$$

which now implies that $H(r)$ is bounded as $r \to \infty$. Since $H(r)$ and $H'(r)$ are positive for $r$ large, we deduce that $H(r) \uparrow \ell > 0$ as $r \to \infty$; and so, $\varphi'(r)^2 + \varphi(r)^2 \to 2\ell > 0$ as $r \to \infty$. Coming back to the equation for $\varphi$, we now see that

$$\varphi'' + \varphi = h\varphi,$$

with $h(r)$ bounded as $r \to \infty$. Multiplying the above equation by $\varphi$ and integrating on $(1, \rho)$, we deduce that

$$\int_1^\rho \varphi'^2 = \int_1^\rho (1-h)\varphi^2 + [\varphi'\varphi]_1^\rho \le C + C\int_1^\rho \varphi^2.$$

Therefore,

$$\int_1^\rho (\varphi'^2 + \varphi^2) \le C + C\int_1^\rho \varphi^2.$$

Since $\liminf \varphi'(r)^2 + \varphi(r)^2 > 0$ as $r \to \infty$, we see that

$$\int_1^\infty \varphi^2 = +\infty,$$

i.e. $u \notin L^2(\mathbb{R}^N)$. In fact, one sees that $u \in L^q(\mathbb{R}^N)$ for $q > 2$ and $u \notin L^q(\mathbb{R}^N)$ for $q \le 2$.

Remark 1.3.10. The proof of Theorems 1.3.1 and 1.3.2 suggests that for every integer $n \ge 0$, there might exist only one initial value $x_n$ such that the solution of (1.3.1) with the initial conditions $u(0) = x_n$ and $u'(0) = 0$ is defined for all $r > 0$, converges to 0 as $r \to \infty$, and has exactly $n$ zeroes on $[0, \infty)$. This uniqueness property was established for $n = 0$ only, and its proof is very delicate (see Kwong [29] and McLeod [37]). It implies in particular uniqueness, up to translations, of positive solutions of the equation $-\triangle u = g(u)$ in $\mathbb{R}^N$ such that $u(x) \to 0$ as $|x| \to \infty$. Indeed, it was shown by Gidas, Ni and Nirenberg [22] that any such solution is spherically symmetric about some point of $\mathbb{R}^N$.

## 1.4. The case of the ball of $\mathbb{R}^N$, $N \ge 2$

In this section, we suppose that $\Omega = B_R = \{x \in \mathbb{R}^N;\ |x| < R\}$ and we look for radial solutions of the equation

$$\begin{cases} -\triangle u = g(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The equation for $u(r) = u(|x|)$ becomes the ODE

$$u'' + \frac{N-1}{r}u' + g(u) = 0, \quad 0 < r < R,$$

with the boundary condition $u(R) = 0$.

It turns out that for the study of such problems, variational methods or super- and subsolutions methods give in many situations more general results. (See Chapters 2 and 3) However, we present below some simple consequences of the results of Section 1.3.

For simplicity, we consider the model case

$$g(u) = -\lambda u + \mu|u|^{p-1}u,$$

and so we look for solutions of the ODE

$$u'' + \frac{N-1}{r}u' - \lambda u + \mu |u|^{p-1}u = 0, \tag{1.4.1}$$

for $0 < r < R$ such that

$$u(R) = 0. \tag{1.4.2}$$

We first apply Proposition 1.3.3, and we obtain the following conclusions.

(i) Suppose $\lambda = 0$, $\mu > 0$ and $(N-2)p \ge N+2$. Then for every $x > 0$, the solution $u$ of (1.4.1) with the initial conditions $u'(0) = 0$ and $u(0) = x$ does *not* satisfy (1.4.2). This follows from property (i) of Proposition 1.3.3. Indeed, if we denote by $\overline{u}$ the solution corresponding to $x = 1$ and $\mu = 1$, then $u(r) = x\overline{u}(x^{\frac{p-1}{2}} r)$.

(ii) Suppose $\lambda = 0$, $\mu > 0$ and $(N-2)p < N+2$. Then for every integer $n \ge 0$, there exists a unique $x_n > 0$ such that the solution $u$ of (1.4.1) with the initial conditions $u'(0) = 0$ and $u(0) = x_n$ satisfies (1.4.2) and has exactly $n$ zeroes on $(0, R)$. This follows from property (ii) of Proposition 1.3.3 and the formula $u(r) = u_0\overline{u}(u_0^{\frac{p-1}{2}} r)$.

(iii) Suppose $\lambda, \mu > 0$ and $(N-2)p < N+2$. Then for every sufficiently large integer $n$, there exists $x_n > 0$ such that the solution $u$ of (1.4.1) with the initial conditions $u'(0) = 0$ and $u(0) = x_n$ satisfies (1.4.2) and has exactly $n$ zeroes on $(0, R)$. Indeed, by scaling, we may assume without loss of generality that $\lambda = \mu = 1$. Next, given any $x > 0$, it follows easily from the proof of Corollary 1.3.5 that the corresponding solution of (1.4.1) oscillates indefinitely. Moreover, it follows easily by continuous dependence that for any integer $k \ge 1$ the $k^{th}$ zero of $u$ depends continuously on $x$. The result now follows from Corollary 1.3.5.

For results in the other cases, see Section 2.7.

CHAPTER 2

# Variational methods

In this chapter, we present the fundamental variational methods that are useful for the resolution of nonlinear PDEs of elliptic type. The reader is referred to Kavian [**28**] and Brezis and Nirenberg [**14**] for a more complete account of variational methods.

## 2.1. Linear elliptic equations

This section is devoted to the basic results of existence of solutions of linear elliptic equations of the form

$$\begin{cases} -\triangle u + au + \lambda u = f & \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \partial\Omega. \end{cases} \tag{2.1.1}$$

Here, $a \in L^\infty(\Omega)$, $\lambda$ is a real parameter and, throughout this section, $\Omega$ is any domain of $\mathbb{R}^N$ (not necessarily bounded nor smooth, unless otherwise specified). We will study a weak formulation of the problem (2.1.1). Given $u \in H^1(\Omega)$, it follows that $-\triangle u + au + \lambda u \in H^{-1}(\Omega)$ (by Proposition 5.1.21), so that the equation (2.1.1) makes sense in $H^{-1}(\Omega)$ for any $f \in H^{-1}(\Omega)$. Taking the $H^{-1} - H^1_0$ duality product of the equation (2.1.1) with any $v \in H^1_0(\Omega)$, we obtain (by formula (5.1.5))

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega auv + \lambda \int_\Omega uv = (f, v)_{H^{-1}, H^1_0}. \tag{2.1.2}$$

Moreover, the boundary condition can be interpreted (in a weak sense) as $u \in H^1_0(\Omega)$. This motivates the following definition.

A weak solution $u$ of (2.1.1) is a function $u \in H^1_0(\Omega)$ that satisfies (2.1.2) for every $v \in H^1_0(\Omega)$. In other words, a weak solution $u$ of (2.1.1) is a function $u \in H^1_0(\Omega)$ such that $-\triangle u + au + \lambda u = f$ in $H^{-1}(\Omega)$. We will often call a weak solution simply a solution.

The simplest tool for the existence and uniqueness of weak solutions of the equation (2.1.1) is Lax-Milgram's lemma.

LEMMA 2.1.1 (Lax-Milgram). *Let $H$ be a Hilbert space and consider a bilinear functional $b : H \times H \to \mathbb{R}$. If there exist $C < \infty$ and $\alpha > 0$ such that*

$$\begin{cases} |b(u,v)| \le C\|u\| \, \|v\|, \ \textit{for all } (u,v) \in H \times H \ \textit{(continuity)}, \\ |b(u,u)| \ge \alpha \|u\|^2, \ \textit{for all } u \in H \ \textit{(coerciveness)}, \end{cases}$$

*then, for every $f \in H^\star$ (the dual space of $H$), the equation*

$$b(u,v) = (f,v)_{H^\star, H} \quad \textit{for all} \quad v \in H, \tag{2.1.3}$$

*has a unique solution $u \in H$.*

PROOF. By the Riesz-Fréchet theorem, there exists $\varphi \in H$ such that

$$(f,v)_{H^\star,H} = (\varphi, v)_H,$$

for all $v \in H$. Furthermore, for any given $u \in H$, the application $v \mapsto b(u, v)$ defines an element of $H^\star$; and so, by the Riesz-Fréchet theorem, there exists an element of $H$, which we denote by $Au$, such that

$$b(u, v) = (Au, v)_H,$$

for all $v \in H$. It is clear that $A : H \to H$ is a linear operator such that

$$\begin{cases} \|Au\|_H \le C\|u\|_H, \\ (Au, u)_H \ge \alpha \|u\|_H^2, \end{cases}$$

for all $u \in H$. We see that (2.1.3) is equivalent to $Au = \varphi$. Given $\rho > 0$, this last equation is equivalent to

$$u = Tu, \tag{2.1.4}$$

where $Tu = u + \rho\varphi - \rho Au$. It is clear that $T : H \to H$ is continuous. Moreover, $Tu - Tv = (u - v) - \rho A(u - v)$; and so,

$$\begin{aligned} \|Tu - Tv\|_H^2 &= \|u - v\|_H^2 + \rho^2 \|A(u - v)\|_H^2 - 2\rho(A(u - v), u - v)_H \\ &\le (1 + \rho^2 C^2 - 2\rho\alpha)\|u - v\|_H^2. \end{aligned}$$

Choosing $\rho > 0$ small enough so that $1 + \rho^2 C^2 - 2\rho\alpha < 1$, $T$ is a strict contraction. By Banach's fixed point theorem, we deduce that $T$ has a unique fixed point $u \in H$, which is the unique solution of (2.1.4). □

In order to study the equation (2.1.1), we make the following definition.
Given $a \in L^\infty(\Omega)$, we set

$$\lambda_1(-\Delta + a; \Omega) = \inf\Big\{ \int_\Omega (|\nabla u|^2 + au^2);\ u \in H_0^1(\Omega), \|u\|_{L^2} = 1 \Big\}. \tag{2.1.5}$$

When there is no risk of confusion, we denote $\lambda_1(-\Delta + a; \Omega)$ by $\lambda_1(-\Delta + a)$ or simply $\lambda_1$.

Remark 2.1.2. Note that $\lambda_1(-\Delta + a; \Omega) \ge -\|a\|_{L^\infty}$. Moreover, it follows from (2.1.5) that

$$\int_\Omega |\nabla u|^2 + \int_\Omega a|u|^2 \ge \lambda_1(-\triangle + a) \int_\Omega |u|^2, \tag{2.1.6}$$

for all $u \in H_0^1(\Omega)$.

When $\Omega$ is bounded, we will see in Section 3.2 that $\lambda_1(-\Delta + a; \Omega)$ is the first eigenvalue of $-\Delta + a$ in $H_0^1(\Omega)$. In the general case, there is the following useful inequality.

Lemma 2.1.3. *Let $a \in L^\infty(\Omega)$ and let $\lambda_1 = \lambda_1(-\Delta + a; \Omega)$ be defined by (2.1.5). Consider $\lambda > -\lambda_1$ and set*

$$\alpha = \min\Big\{1, \frac{\lambda + \lambda_1}{1 + \lambda_1 + \|a\|_{L^\infty}}\Big\} > 0, \tag{2.1.7}$$

*by Remark 2.1.2. It follows that*

$$\int_\Omega |\nabla u|^2 + \int_\Omega au^2 + \lambda \int_\Omega u^2 \ge \alpha \|u\|_{H^1}^2, \tag{2.1.8}$$

*for all $u \in H_0^1(\Omega)$.*

PROOF. We denote by $\Phi(u)$ the left-hand side of (2.1.8). It follows from (2.1.6) that, given any $0 \le \varepsilon \le 1$,

$$\begin{aligned}\Phi(u)^2 &\ge \varepsilon \int_\Omega (|\nabla u|^2 + a|u|^2) + ((1-\varepsilon)\lambda_1 + \lambda) \int_\Omega |u|^2 \\ &\ge \varepsilon \int_\Omega |\nabla u|^2 + ((1-\varepsilon)\lambda_1 + \lambda - \varepsilon \|a\|_{L^\infty}) \int_\Omega |u|^2 \\ &= \varepsilon \int_\Omega |\nabla u|^2 + (\lambda + \lambda_1 - \varepsilon(\lambda_1 + \|a\|_{L^\infty})) \int_\Omega |u|^2.\end{aligned}$$

The result follows by letting $\varepsilon = \alpha$. □

Our main result of this section is the following existence and uniqueness result.

THEOREM 2.1.4. *Let $a \in L^\infty(\Omega)$ and let $\lambda_1 = \lambda_1(-\Delta + a; \Omega)$ be defined by (2.1.5). If $\lambda > -\lambda_1$, then for every $f \in H^{-1}(\Omega)$, the equation (2.1.1) has a unique weak solution. In addition,*

$$\alpha \|u\|_{H^1} \le \|f\|_{H^{-1}} \le (1 + \|a\|_{L^\infty} + |\lambda|)\|u\|_{H^1}, \tag{2.1.9}$$

*where $\alpha$ is defined by (2.1.7). In particular, the mapping $f \mapsto u$ is an isomorphism $H^{-1}(\Omega) \to H^1_0(\Omega)$.*

PROOF. Let

$$b(u, v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega auv + \lambda \int_\Omega uv,$$

for $u, v \in H^1_0(\Omega)$. It is clear that $b$ is continuous, and it follows from (2.1.8) that $b$ is coercive. Existence and uniqueness now follow by applying Lax-Milgram's lemma in $H = H^1_0(\Omega)$ with $b$ defined above. Next, we deduce from (2.1.8) that

$$\alpha \|u\|^2_{H^1} \le b(u, u) = (f, u)_{H^{-1}, H^1_0} \le \|f\|_{H^{-1}} \|u\|_{H^1},$$

from which we obtain the left-hand side of (2.1.9). Finally,

$$\|f\|_{H^{-1}} \le \|\Delta u\|_{H^{-1}} + \|au\|_{H^{-1}} + |\lambda|\, \|u\|_{H^{-1}} \le (1 + \|a\|_{L^\infty} + |\lambda|)\|u\|_{H^1},$$

which proves the right-hand side of (2.1.9). □

REMARK 2.1.5. If $a = 0$, then $\lambda_1 = \lambda_1(-\Delta; \Omega)$ depends only on $\Omega$. $\lambda_1$ may equal 0 or be positive. The property $\lambda_1 > 0$ is equivalent to Poincaré's inequality. In particular, if $\Omega$ has finite measure, then $\lambda_1 > 0$ by Theorem 5.4.19. On the other hand, one verifies easily that if $\Omega = \mathbb{R}^N$, then $\lambda_1 = 0$ (Take for example $u_\varepsilon(x) = \varepsilon^{\frac{N}{2}} \varphi(\varepsilon x)$ with $\varphi \in C^\infty_c(\mathbb{R}^N)$, $\varphi \not\equiv 0$ and let $\varepsilon \downarrow 0$). If $\Omega = \mathbb{R}^N \setminus K$, where $K$ is a compact subset of $\mathbb{R}^N$, a similar argument (translate $u_\varepsilon$ in such a way that $\operatorname{supp} u_\varepsilon \subset \Omega$) shows that as well $\lambda_1 = 0$.

REMARK 2.1.6. The assumption $\lambda > -\lambda_1$ implies the existence of a solution of (2.1.1) for all $f \in H^{-1}(\Omega)$. However, this condition may be necessary or not, depending on $\Omega$. Let us consider several examples to illustrate this fact.

(i) Suppose $\Omega$ is bounded. Let $(\lambda_n)_{n \ge 1}$ be the sequence of eigenvalues of $-\triangle + a$ in $H^1_0(\Omega)$ (see Section 3.2) and let $(\varphi_n)_{n \ge 1}$ be a corresponding orthonormal system of eigenvectors. Given $f \in H^{-1}(\Omega)$, we may write $f = \sum_{n \ge 1} \alpha_n \varphi_n$ with $\sum \lambda_n^{-1} |\alpha_n|^2 < \infty$. A function $u \in H^1_0(\Omega)$ is given by $u = \sum_{n \ge 1} a_n \varphi_n$ with $\sum \lambda_n |a_n|^2 < \infty$. Since necessarily $(\lambda_n + \lambda) a_n = \alpha_n$ for a solution of (2.1.1), we see that if $\lambda \ne -\lambda_n$ for all $n \ge 1$, then (2.1.1) has a solution for all $f \in H^{-1}(\Omega)$. On the other hand, if $\lambda = -\lambda_n$ for some $n \ge 1$, then it is clear that for $f = \varphi_n$ the equation (2.1.1) does not have any solution. So in this case, the equation (2.1.1) has a weak solution for all $f \in H^{-1}(\Omega)$ if and only if $\lambda \ne -\lambda_n$ for all $n \ge 1$.

(ii) Suppose $\Omega = \mathbb{R}^N$ and let $a = 0$, so that in particular $\lambda_1 = 0$. We claim that there exists $f \in H^{-1}(\mathbb{R}^N)$ such that for any $\lambda \le 0$, the equation (2.1.1) does not have any solution. Indeed, suppose $\lambda \le 0$ and consider $f(x) = e^{-|x|^2}$. We have $\widehat{f}(\xi) = \pi^{\frac{N}{2}} e^{-\pi^2|\xi|^2}$. If (2.1.1) has a solution $u$, then by applying the Fourier transform, we obtain $(4\pi^2|\xi|^2 + \lambda)\widehat{u}(\xi) = \widehat{f}(\xi) = \pi^{\frac{N}{2}} e^{-\pi^2|\xi|^2}$, thus $\widehat{u}(\xi) = \pi^{\frac{N}{2}} e^{-\pi^2|\xi|^2}(4\pi^2|\xi|^2 + \lambda)^{-1} \notin L^2(\mathbb{R}^N)$. This yields a contradiction.

(iii) Suppose $N \ge 2$ and $\Omega = \mathbb{R} \times \omega$, where $\omega$ is a bounded, open domain of $\mathbb{R}^{N-1}$, and let $a = 0$. We claim that there exists $f \in H^{-1}(\Omega)$ such that for any $\lambda \le -\lambda_1$, the equation (2.1.1) does not have any solution. Indeed, let $(\widetilde{\lambda}_n)_{n\ge1}$ be the sequence of eigenvalues of $-\triangle$ in $H^1_0(\omega)$ and let $(\widetilde{\varphi}_n)_{n\ge1}$ be a corresponding orthonormal system of eigenvectors (see Section 3.2 below). It is not difficult to verify that $\lambda_1 = \widetilde{\lambda}_1$. Consider $f(x,y) = e^{-|x|^2}\widetilde{\varphi}_1(y)$ for $(x,y) \in \mathbb{R} \times \omega$. If (2.1.1) has a solution $u$, we obtain that $v(\xi, y)$, the Fourier transform of $u(x,y)$ in the variable $x$, has the form $v(\xi,y) = \theta(\xi)\widetilde{\varphi}_1(y)$ with $(4\pi^2|x|^2 + \widetilde{\lambda}_1 + \lambda)\theta(\xi) = \pi^{\frac{1}{2}} e^{-\pi^2|\xi|^2}$. If $\lambda < -\widetilde{\lambda}_1 = -\lambda_1$, then $\theta(\cdot) \notin L^2(\mathbb{R})$, thus $u \notin L^2(\Omega)$, which is absurd.

## 2.2. $C^1$ functionals

We begin by recalling some definitions. Let $X$ be a Banach space and consider a functional $F \in C(X, \mathbb{R})$. $F$ is (Fréchet) differentiable at some point $x \in X$ if there exists $L \in X^\star$ such that

$$\frac{|F(x+y) - F(x) - (L,y)_{X^\star,X}|}{\|y\|} \underset{\|y\|\downarrow 0}{\longrightarrow} 0.$$

Such a $L$ is then unique, is called the derivative of $F$ at $X$ and is denoted $F'(x)$. $F \in C^1(X, \mathbb{R})$ if $F$ is differentiable at all $x \in X$ and if the mapping $x \mapsto F'(x)$ is continuous $X \to X^\star$.

There is a weaker notion of derivative, the Gâteaux derivative. A functional $F \in C(X, \mathbb{R})$ is Gâteaux-differentiable at some point $x \in X$ if there exists $L \in X^\star$ such that

$$\frac{F(x+ty) - F(x)}{t} \underset{t\downarrow 0}{\longrightarrow} (L,y)_{X^\star,X},$$

for all $y \in X$. Such a $L$ is then unique, is called the Gâteaux-derivative of $F$ at $X$ and is denoted $F'(x)$. It is clear that if a functional is Fréchet-differentiable at some $x \in X$, then it is also Gâteaux-differentiable and both derivatives agree. On the other hand, there exist functionals that are Gâteaux-differentiable at some point where they are not Fréchet-differentiable. However, it is well-know that if a functional $F \in C(X, \mathbb{R})$ is Gâteaux-differentiable at every point $x \in X$, and if its Gâteaux derivative $F'(x)$ is continuous $X \to X^\star$, then $F \in C^1(X, \mathbb{R})$. In other words, in order to show that $F$ is $C^1$, we need only show that $F$ is Gâteaux-differentiable at every point $x \in X$, and that $F'(x)$ is continuous $X \to X^\star$.

We now give several examples of functionals arising in PDEs and which are $C^1$ in appropriate Banach spaces. In what follows, $\Omega$ is an arbitrary domain of $\mathbb{R}^N$.

Consider a function $g \in C(\mathbb{R}, \mathbb{R})$, and assume that there exist $1 \le r < \infty$ and a constant $C$ such that

$$|g(u)| \le C|u|^r, \tag{2.2.1}$$

for all $u \in \mathbb{R}$. Setting

$$G(u) = \int_0^u g(s)\, ds, \tag{2.2.2}$$

it follows that $|G(u)| \le \dfrac{C}{r+1}|u|^{r+1}$. Therefore, we may define

$$J(u) = \int_\Omega G(u(x))\,dx, \tag{2.2.3}$$

for all $u \in L^{r+1}(\Omega)$. Our first result is the following.

PROPOSITION 2.2.1. *Assume $g \in C(\mathbb{R}, \mathbb{R})$ satisfies (2.2.1) for some $r \in [1, \infty)$, let $G$ be defined by (2.2.2) and let $J$ be defined by (2.2.3). It follows that the mapping $u \mapsto g(u)$ is continuous from $L^{r+1}(\Omega)$ to $L^{\frac{r+1}{r}}(\Omega)$. Moreover, $J \in C^1(L^{r+1}(\Omega), \mathbb{R})$ and*

$$J'(u) = g(u), \tag{2.2.4}$$

*for all $u \in L^{r+1}(\Omega)$.*

PROOF. It is clear that $\|g(u)\|_{L^{\frac{r+1}{r}}} \le C\|u\|_{L^{r+1}}^{r+1}$, thus $g$ maps $L^{r+1}(\Omega)$ to $L^{\frac{r+1}{r}}(\Omega)$. We now show that $g$ is continuous. Assume by contradiction that $u_n \to u$ in $L^{r+1}(\Omega)$ as $n \to \infty$ and that $\|g(u_n) - g(u)\|_{L^{\frac{r+1}{r}}} \ge \varepsilon > 0$. By possibly extracting a subsequence, we may assume that $u_n \to u$ a.e.; and so, $g(u_n) \to g(u)$ a.e. Furthermore, we may also assume that there exists $f \in L^{r+1}(\Omega)$ such that $|u_n| \le f$ a.e. Applying (2.2.1) and the dominated convergence theorem, we deduce that $g(u_n) \to g(u)$ in $L^{\frac{r+1}{r}}(\Omega)$. Contradiction.

Consider now $u, v \in L^{r+1}(\Omega)$. Since $g = G'$, we see that

$$\frac{G(u+tv) - G(u)}{t} - g(u)v \underset{t\downarrow 0}{\longrightarrow} 0,$$

a.e. Note that by (2.2.1), $|g(u)v| \le C|u|^r|v| \in L^1(\Omega)$ and for $0 < t < 1$

$$\begin{aligned}\frac{|G(u+tv) - G(u)|}{t} &\le \frac{1}{t}\Big|\int_u^{u+tv} g(s)\,ds\Big| \le C|v|(|u|^r + t^r|v|^r) \\ &\le C|v|(|u|^r + |v|^r) \in L^1(\Omega).\end{aligned}$$

By dominated convergence, we deduce that

$$\int_\Omega \Big|\frac{G(u+tv) - G(u)}{t} - g(u)v\Big| \underset{t\downarrow 0}{\longrightarrow} 0.$$

This means that $J$ is Gâteaux differentiable at $u$ and that $J'(u) = g(v)$. Since $g$ is continuous $L^{r+1}(\Omega) \to L^{\frac{r+1}{r}}(\Omega)$, the result follows. □

Consider again a function $g \in C(\mathbb{R}, \mathbb{R})$, and assume now that there exist $1 \le r < \infty$ and a constant $C$ such that

$$|g(u)| \le C(|u| + |u|^r), \tag{2.2.5}$$

for all $u \in \mathbb{R}$. (Note that in particular $g(0) = 0$.) Consider $G$ defined by (2.2.2) and, given $h_1 \in H^{-1}(\Omega)$ and $h_2 \in L^{\frac{r+1}{r}}(\Omega)$, let

$$\begin{aligned}J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(u)& \\ - (h_1, u)_{H^{-1}, H_0^1} - (h_2, u)_{L^{\frac{r+1}{r}}, L^{r+1}},&\end{aligned} \tag{2.2.6}$$

for $u \in H_0^1(\Omega) \cap L^{r+1}(\Omega)$. We note that $G(u) \in L^1(\Omega)$, so $J$ is well defined. Let

$$X = H_0^1(\Omega) \cap L^{r+1}(\Omega), \tag{2.2.7}$$

and set

$$\|u\|_X = \|u\|_{H^1} + \|u\|_{L^{r+1}}, \tag{2.2.8}$$

for $u \in X$. It follows immediately that $X$ is a Banach space with the norm $\|\cdot\|_X$. One can show that $X^\star = H^{-1}(\Omega) + L^{\frac{r+1}{r}}(\Omega)$, where the Banach space $H^{-1}(\Omega) +$

$L^{\frac{r+1}{r}}(\Omega)$ is defined appropriately (see Bergh and Löfström [10], Lemma 2.3.1 and Theorem 2.7.1). We will not use that property, whose proof is rather delicate, but we will use the simpler properties $H^{-1}(\Omega) \hookrightarrow X^\star$ and $L^{\frac{r+1}{r}}(\Omega) \hookrightarrow X^\star$. This is immediate since, given $f \in H^{-1}(\Omega)$, the mapping $u \mapsto (f,u)_{H^{-1},H_0^1}$ defines clearly an element of $X^\star$. Furthermore, this defines an injection because if $(f,u)_{H^{-1},H_0^1}$ for all $u \in X$, then in particular $(f,u)_{H^{-1},H_0^1}$ for all $u \in C_c^\infty(\Omega)$. By density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, we deduce $f = 0$. A similar argument shows that $L^{\frac{r+1}{r}}(\Omega) \hookrightarrow X^\star$.

Corollary 2.2.2. *Assume that $g \in C(\mathbb{R},\mathbb{R})$ satisfies (2.2.5) and let $h_1 \in H^{-1}(\Omega)$ and $h_2 \in L^{\frac{r+1}{r}}(\Omega)$. Let $J$ be defined by (2.2.6) and let $X$ be defined by (2.2.7)-(2.2.8). Then $g$ is continuous $X \to X^\star$, $J \in C^1(X,\mathbb{R})$ and*

$$J'(u) = -\triangle u - g(u) - h_1 - h_2, \tag{2.2.9}$$

*for all $u \in X$.*

Proof. We first show that $g$ is continuous $X \to X^\star$, and for that we split $g$ in two parts. Namely, we set

$$g(u) = g_1(u) + g_2(u),$$

where $g_1(u) = g(u)$ for $|u| \le 1$ and $g_1(u) = 0$ for $|u| \ge 2$. It follows immediately that

$$|g_1(u)| \le C|u|,$$

and that

$$|g_2(u)| \le C|u|^r,$$

by possibly modifying the value of $C$. By Proposition 2.2.1, we see that the mapping $u \mapsto g_1(u)$ is continuous $L^2(\Omega) \to L^2(\Omega)$, hence $H_0^1(\Omega) \to H^{-1}(\Omega)$, hence $X \to X^\star$. As well, the mapping $u \mapsto g_2(u)$ is continuous $L^{r+1}(\Omega) \to L^{\frac{r+1}{r}}(\Omega)$, hence $X \to X^\star$. Therefore, $g = g_1 + g_2$ is continuous $X \to X^\star$.

We now define

$$\widetilde{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2,$$

so that $\widetilde{J} \in C^1(H_0^1(\Omega),\mathbb{R}) \subset C^1(X,\mathbb{R})$ and $\widetilde{J}'(u) = -\triangle u$ (see Corollary 5.1.22). Next, let

$$J_0(u) = (h_1,u)_{H^{-1},H_0^1} + (h_2,u)_{L^{\frac{r+1}{r}},L^{r+1}} := J_0^1(u) + J_0^2(u).$$

One verifies easily that $J_0^1 \in C^1(H_0^1(\Omega),\mathbb{R})$ and that $J_0^1{}'(u) = h_1$. Also, $J_0^2 \in C^1(L^{r+1},\mathbb{R})$ and that $J_0^2{}'(u) = h_2$. Thus $J_0 \in C^1(X,\mathbb{R})$ and $J_0'(u) = h_1 + h_2$. Finally, let

$$J_\ell(u) = \int_\Omega G_\ell(u),$$

for $\ell = 1,2$, where $G_\ell(u) = \displaystyle\int_0^u g_\ell(s)\,ds$. The result now follows by applying Proposition 2.2.1 to the functionals $J_\ell$ and writing $J = \widetilde{J} - J_0 - J_1 - J_2$. □

Corollary 2.2.3. *Assume that $g \in C(\mathbb{R},\mathbb{R})$ satisfies (2.2.5), with the additional assumption $(N-2)r \le N+2$, and let $h \in H^{-1}(\Omega)$. Let $J$ be defined by (2.2.6) (with $h_1 = h$ and $h_2 = 0$). Then $g$ is continuous $H_0^1(\Omega) \to H^{-1}(\Omega)$, $J \in C^1(H_0^1(\Omega),\mathbb{R})$ and (2.2.9) holds for all $u \in H_0^1(\Omega)$.*

Proof. Since $H_0^1(\Omega) \cap L^{r+1}(\Omega) = H_0^1(\Omega)$ by Sobolev's embedding theorem, the result follows from Corollary 2.2.2. □

## 2.3. Global minimization

We begin by recalling some simple properties. Let $X$ be a Banach space and consider a functional $F \in C^1(X, \mathbb{R})$. A critical point of $F$ is an element $x \in X$ such that $F'(x) = 0$. If $F$ achieves its minumum, i.e. if there exists $x_0 \in X$ such that

$$F(x_0) = \inf_{x \in X} F(x),$$

then $x_0$ is a critical point of $F$. Indeed, if $F'(x_0) \neq 0$, then there exists $y \in X$ such that $(F'(x_0), y)_{X^\star, X} < 0$. It follows from the definition of the derivative that $F(x_0 + ty) \leq F(x_0) + \frac{t}{2}(F'(x_0), y)_{X^\star, X} < F(x_0)$ for $t > 0$ small enough, which is absurd.

In this section, we will construct solutions of the equation

$$\begin{cases} -\triangle u = g(u) + h & \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \partial\Omega, \end{cases} \tag{2.3.1}$$

by minimizing a functional $J$ such that $J'(u) = -\triangle u - g(u) - h$ in an appropriate Banach space. Of course, this will require assumptions on $g$ and $h$. We begin with the following result.

THEOREM 2.3.1. *Assume that $g \in C(\mathbb{R}, \mathbb{R})$ satisfies (2.2.5), with the additional assumption $(N-2)r \leq N+2$. Let $\lambda_1 = \lambda_1(-\Delta)$ be defined by (2.1.5), and suppose further that*

$$G(u) \leq -\frac{\lambda}{2} u^2, \tag{2.3.2}$$

*for all $u \in \mathbb{R}$, with $\lambda > -\lambda_1$. (Here, $G$ is defined by (2.2.2).) Finally, let $h \in H^{-1}(\Omega)$ and let $J$ be defined by (2.2.6) with $h_1 = h$ and $h_2 = 0$ (so that $J \in C^1(H_0^1(\Omega), \mathbb{R})$ by Corollary 2.2.3). Then there exists $u \in H_0^1(\Omega)$ such that*

$$J(u) = \inf_{v \in H_0^1(\Omega)} J(v).$$

*In particular, $u$ is a weak solution of (2.3.1) in the sense that $u \in H_0^1(\Omega)$ and $-\triangle u = g(u) + h$ in $H^{-1}(\Omega)$.*

For the proof of Theorem 2.3.1, we will use the following lemma.

LEMMA 2.3.2. *Let $\lambda > -\lambda_1$, where $\lambda_1 = \lambda_1(-\Delta)$ is defined by (2.1.5). Let $h \in H^{-1}(\Omega)$ and set*

$$\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2 + (h, u)_{H^{-1}, H_0^1},$$

*for all $u \in H_0^1(\Omega)$. If $(u_n)_{n \geq 0}$ is a bounded sequence of $H_0^1(\Omega)$, then there exist a subsequence $(u_{n_k})_{k \geq 0}$ and $u \in H_0^1(\Omega)$ such that*

$$\Psi(u) \leq \liminf_{k \to \infty} \Psi(u_{n_k}), \tag{2.3.3}$$

*and $u_{n_k} \underset{k \to \infty}{\longrightarrow} u$ a.e. in $\Omega$.*

PROOF. Since $(u_n)_{n \geq 0}$ is a bounded sequence of $H_0^1(\Omega)$, there exist $u \in H_0^1(\Omega)$ and a subsequence $(u_{n_k})_{k \geq 0}$ such that $u_{n_k} \to u$ a.e. in $\Omega$ as $k \to \infty$ and $u_{n_k} \to u$ in $L^2(\Omega \cap \{|x| < R\})$ for all $R > 0$ (see Remark 5.5.6). For proving (2.3.3), we proceed in two steps.

STEP 1. For every $f \in H^{-1}(\Omega)$,

$$(f, u_{n_k})_{H^{-1}, H_0^1} \underset{k \to \infty}{\longrightarrow} (f, u)_{H^{-1}, H_0^1}.$$

Indeed,

$$(f, u_{n_k} - u)_{H^{-1}, H_0^1} \underset{k \to \infty}{\longrightarrow} 0,$$

when $f \in C_c(\Omega)$, by local $L^2$ convergence. The result follows by density of $C_c(\Omega)$ in $H^{-1}(\Omega)$ (see Proposition 5.1.18).

STEP 2. Conclusion. By Step 1, we need only show that if

$$\Phi(u) = \int_\Omega |\nabla u|^2 + \lambda \int_\Omega u^2,$$

then $\Phi(u) \le \liminf \Phi(u_{n_k})$ as $k \to \infty$. Indeed, we have $\Phi(u) \ge \alpha \|u\|_{H^1}^2$ by (2.1.8). Since clearly $\Phi(u) \le \max\{1, \lambda\} \|u\|_{H^1}^2$, it follows that

$$|||v||| = \Phi(v)^{\frac{1}{2}},$$

defines an equivalent norm on $H_0^1(\Omega)$. We equip $H^{-1}(\Omega)$ with the corresponding dual norm $|||\cdot|||_\star$. (Note that this dual norm is equivalent to the original one and that, by definition, the duality product $(\cdot\,,\,\cdot)_{H^{-1},H_0^1}$ is unchanged.) We have

$$|||u||| = \sup\{(f,u)_{H^{-1},H_0^1};\ f \in H^{-1}(\Omega), |||f|||_\star = 1\}.$$

By Step 1,

$$(f,u)_{H^{-1},H_0^1} = \lim_{k\to\infty} (f, u_{n_k})_{H^{-1},H_0^1},$$

for every $f \in H^{-1}(\Omega)$. Since $(f, u_{n_k})_{H^{-1},H_0^1} \le |||f|||_\star |||u_{n_k}|||$, we deduce that

$$(f,u)_{H^{-1},H_0^1} \le |||f|||_\star \liminf_{k\to\infty} |||u_{n_k}|||,$$

from which the result follows. □

PROOF OF THEOREM 2.3.1. We first note that, by (2.3.2),

$$J(u) \ge \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2 - (h,u)_{H^{-1},H_0^1}.$$

By (2.1.8), this implies that

$$J(u) \ge \frac{\alpha}{2}\|u\|_{H^1}^2 - \|h\|_{H^{-1}}\|u\|_{H^1} \ge \frac{\alpha}{4}\|u\|_{H^1}^2 - \frac{1}{\alpha}\|h\|_{H^{-1}}^2, \tag{2.3.4}$$

for all $u \in H_0^1(\Omega)$, where $\alpha$ is defined by (2.1.7). It follows from (2.3.4) that $J$ is bounded from below. Let

$$m = \inf_{v \in H_0^1(\Omega)} J(v) > -\infty,$$

and let $(u_n)_{n\ge 0} \subset H_0^1(\Omega)$ be a minimizing sequence. It follows in particular from (2.3.4) that $(u_n)_{n\ge0}$ is bounded in $H_0^1(\Omega)$. We now write

$$J(u) = J_1(u) + J_2(u),$$

where

$$J_1(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2 - (h,u)_{H^{-1},H_0^1},$$

and

$$J_2(u) = \int_\Omega \Bigl(-G(u) - \frac{\lambda}{2}u^2\Bigr).$$

Applying Lemma 2.3.2, we find that there exist $u \in H_0^1(\Omega)$ and a subsequence $(u_{n_k})_{k\ge0}$ such that $u_{n_k} \to u$ a.e. in $\Omega$ as $k \to \infty$ and

$$J_1(u) \le \liminf_{k\to\infty} J_1(u_{n_k}).$$

Since $-G(t) - \dfrac{\lambda}{2}t^2 \ge 0$ by (2.3.2), it follows from Fatou's lemma that

$$J_2(u) \le \liminf_{k\to\infty} J_2(u_{n_k});$$

and so, $J(u) \le \liminf J(u_{n_k}) = m$ as $k \to \infty$. Therefore, $J(u) = m$, which proves the first part of the result. Finally, we have $J'(u) = 0$, i.e. $-\triangle u = g(u) + h$ by Corollary 2.2.3. $\square$

REMARK 2.3.3. If $\Omega$ is bounded, then one can weaken the assumption (2.3.2). One may assume instead that (2.3.2) holds for $|u|$ large enough. Indeed, we have then

$$G(u) \le C - \frac{\lambda}{2} u^2,$$

for all $u \in \mathbb{R}$ and some constant $C$. The construction of the minimizing sequence is made as above, since one obtains instead of (2.3.4)

$$J(u) \ge \frac{\alpha}{2} \|u\|_{H^1}^2 - \|h\|_{H^{-1}} \|u\|_{H^1} - C|\Omega| \ge \frac{\alpha}{4} \|u\|_{H^1}^2 - \frac{1}{\alpha} \|h\|_{H^{-1}}^2 - C|\Omega|.$$

For the passage to the limit, we have $G(u) \le \mu u^2$ for $\mu$ large enough (since $G(u) = O(u^2)$ near 0). Therefore, one can use the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ (Theorem 5.5.5) to pass to the limit in the negative part of $J$.

REMARK 2.3.4. We give below some applications of Theorem 2.3.1 and Remark 2.3.3.

(i) If $\Omega$ is a bounded subset, then Theorem 2.3.1 (together with Remark 2.3.3 above) applies for example to the equation

$$-\triangle u + \lambda u + a|u|^{p-1}u - b|u|^{q-1}u = f,$$

where $f \in H^{-1}(\Omega)$ (for example, $f$ may be a constant), $\lambda \in \mathbb{R}$, $a > 0$, $b \in \mathbb{R}$ and $1 < q < p \le (N+2)/(N-2)$.

(ii) When $\Omega$ is not bounded, Theorem 2.3.1 applies to the same equation with the additional restrictions $\lambda > 0$ and $b < \lambda^{\frac{p-q}{p-1}} a^{\frac{q-1}{p-1}} (p-1)(q+1)(p-q)^{-\frac{p-q}{p-1}} ((q-1)(p+1))^{-\frac{q-1}{p-1}}$.

In the examples of Remark 2.3.4 (i) and (ii), one can indeed remove the assumption $p \le (N+2)/(N-2)$. More generally, one can remove this assumption in Theorem 2.3.1, provided one assumes a stronger upper bound on $G$. This is the object of the following result.

THEOREM 2.3.5. *Assume that $g \in C(\mathbb{R}, \mathbb{R})$ satisfies (2.2.5) for some $r \ge 1$. Let $\lambda_1 = \lambda_1(-\Delta)$ be defined by (2.1.5), and suppose that $G$ satisfies (2.3.2) for all $u \in \mathbb{R}$, with $\lambda > -\lambda_1$. Suppose further that*

$$G(u) \le -a|u|^{r+1}, \tag{2.3.5}$$

*for all $|u| \ge M$, where $a > 0$. Finally, let $h_1 \in H^{-1}(\Omega)$ and $h_2 \in L^{\frac{r+1}{r}}(\Omega)$ and let $J$ be defined by (2.2.6) (so that $J \in C^1(H_0^1(\Omega) \cap L^{r+1}(\Omega), \mathbb{R})$ by Corollary 2.2.2). Then there exists $u \in H_0^1(\Omega) \cap L^{r+1}(\Omega)$ such that*

$$J(u) = \inf_{v \in H_0^1(\Omega)} J(v).$$

*In particular, $u$ is a weak solution of (2.3.1) with $h = h_1 + h_2$ in the sense that $u \in H_0^1(\Omega) \cap L^{r+1}(\Omega)$ and $-\triangle u = g(u) + h_1 + h_2$ in $(H_0^1(\Omega) \cap L^{r+1}(\Omega))^\star$.*

PROOF. The proof is parallel to the proof of Theorem 2.3.1. We first observe that by (2.3.2) and (2.3.5) we have

$$G(u) \le -\frac{\lambda}{2} u^2 - a|u|^{r+1}, \tag{2.3.6}$$

for all $u \in \mathbb{R}$, by possibly modifying $a > 0$ and $\lambda > -\lambda_1$. It follows from (2.3.6) that

$$J(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2 + a\int_\Omega |u|^{r+1} - (h_1, u)_{H^{-1},H^1_0} - (h_2, u)_{L^{\frac{r+1}{r}},L^{r+1}}.$$

By (2.1.8), this implies that

$$J(u) \geq \frac{\alpha}{2}\|u\|_{H^1}^2 + a\|u\|_{L^{r+1}}^{r+1} - \|h_1\|_{H^{-1}}\|u\|_{H^1} - \|h_2\|_{L^{\frac{r+1}{r}}}\|u\|_{L^{r+1}},$$

so that

$$J(u) \geq \frac{\alpha}{4}\|u\|_{H^1}^2 + \frac{a}{2}\|u\|_{L^{r+1}}^{r+1} - \frac{1}{\alpha}\|h\|_{H^{-1}}^2 - \frac{2^{\frac{1}{r}} r}{a^{\frac{1}{r}}(r+1)^{\frac{r+1}{r}}}\|h_2\|_{L^{\frac{r+1}{r}}}^{\frac{r+1}{r}}, \tag{2.3.7}$$

for all $u \in H^1_0(\Omega) \cap L^{r+1}(\Omega)$, where $\alpha$ is defined by (2.1.7). It follows from (2.3.7) that $J$ is bounded from below on $H^1_0(\Omega) \cap L^{r+1}(\Omega)$. Let

$$m = \inf_{v \in H^1_0 \cap L^{r+1}} J(v) > -\infty,$$

and let $(u_n)_{n\geq 0} \subset H^1_0(\Omega) \cap L^{r+1}(\Omega)$ be a minimizing sequence. It follows in particular from (2.3.7) that $(u_n)_{n\geq 0}$ is bounded in $H^1_0(\Omega) \cap L^{r+1}(\Omega)$. We now write

$$J(u) = J_1(u) + J_2(u) + J_3(u),$$

where

$$J_1(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2 - (h_1, u)_{H^{-1},H^1_0},$$

$$J_2(u) = \int_\Omega \Big(-G(u) - \frac{\lambda}{2}u^2\Big),$$

and

$$J_3(u) = (h_2, u)_{L^{\frac{r+1}{r}},L^{r+1}}.$$

Applying Lemma 2.3.2, we find that there exist $u \in H^1_0(\Omega)$ and a subsequence $(u_{n_k})_{k\geq 0}$ such that $u_{n_k} \to u$ a.e. in $\Omega$ as $k \to \infty$ and

$$J_1(u) \leq \liminf_{k\to\infty} J_1(u_{n_k}).$$

Since $-G(t) - \lambda t^2/2 \geq 0$ by (2.3.2), it follows from Fatou's lemma that

$$J_2(u) \leq \liminf_{k\to\infty} J_2(u_{n_k}).$$

Applying Corollary 5.5.2 and Lemma 5.5.3, we may also assume, after possibly extracting a subsequence, that

$$(h_2, u_{n_k})_{L^{\frac{r+1}{r}},L^{r+1}} \underset{k\to\infty}{\longrightarrow} (h_2, u)_{L^{\frac{r+1}{r}},L^{r+1}};$$

and so, $J(u) \leq \liminf J(u_{n_k}) = m$ as $k \to \infty$. Therefore, $J(u) = m$, which proves the first part of the result. Finally, we have $J'(u) = 0$, i.e. $-\triangle u = g(u) + h$ by Corollary 2.2.2. □

Remark 2.3.6. Here are some comments on Theorem 2.3.5.

(i) If $\Omega$ is bounded, then one does not need the assumption (2.3.2). (See Remark 2.3.3 for the necessary modifications to the proof.)

(ii) One may apply Theorem 2.3.5 (along with (i) above) to the examples of Remark 2.3.4, but without the restriction $p \leq (N+2)/(N-2)$.

Let us observe that the equation (2.3.1) may have one or several solutions, depending on $g$ and $h$. For example, if $h = 0$ and $g(0) = 0$, then $u = 0$ is a trivial solution. It may happen that there are more solutions. In that case, we speak of nontrivial solutions. We give below two examples that illustrate the two different situations.

THEOREM 2.3.7. *Let $g$, $\lambda$ and $h$ be as in Theorem 2.3.1, and let $u$ be the solution of (2.3.1) given by Theorem 2.3.1. If the mapping $u \mapsto g(u) + \lambda u$ is nonincreasing, then $u$ is the unique solution in $H_0^1(\Omega)$ of (2.3.1).*

PROOF. We write $J(u) = J_0(u) + J_1(u) + J_2(u)$ with

$$J_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2,$$
$$J_1(u) = \int_\Omega \left(-G(u) - \frac{\lambda}{2}u^2\right),$$

and

$$J_2(u) = (h, u)_{H^{-1},H_0^1}.$$

We observe that, since $\lambda > -\lambda_1$, $J_0(u)$ is strictly convex. (Indeed, if $a(u, v)$ is a bilinear functional such that $a(u, u) \geq 0$, then the mapping $u \mapsto a(u, u)$ is convex; and if $a(u, u) > 0$ for all $u \neq 0$, then it is strictly convex.) Furthermore, $J_1$ is convex because the mapping $u \mapsto -g(u) - \lambda u$ is nondecreasing. Finally, $J_2$ is linear, thus convex. Therefore, $J$ is strictly convex. Assume now $u$ and $v$ are two solutions, so that $J'(u) = J'(v) = 0$. It follows that $(J'(u) - J'(v), u - v)_{H^{-1},H_0^1} = 0$, and since $J$ is strictly convex, this implies $u = v$. □

REMARK 2.3.8. One shows similarly that, under the assumptions of Theorem 2.3.5, and if the mapping $u \mapsto g(u) + \lambda u$ is nonincreasing, then the solution of (2.3.1) is unique in $H_0^1(\Omega) \cap L^{r+1}(\Omega)$. Note also that if $\lambda = -\lambda_1$, then the same conclusion holds, provided the mapping $u \mapsto g(u) + \lambda u$ is decreasing. Indeed, in this case, $J_0$ is not strictly convex (but still convex), but $J_1$ is strictly convex.

The above results apply for example to the equation $-\triangle u + \lambda u + a|u|^{p-1}u = h_1 + h_2$, with $\lambda > -\lambda_1$, $a > 0$, $p > 1$, $h_1 \in H^{-1}(\Omega)$ and $h_2 \in H^{-1}(\Omega) \cap L^{\frac{p+1}{p}}(\Omega)$. In the case $\lambda < -\lambda_1$, the situation is quite different, as shows the following result.

THEOREM 2.3.9. *Let $\Omega$ be a bounded domain of $\mathbb{R}^N$, and assume $\lambda < -\lambda_1$ where $\lambda_1 = \lambda_1(-\Delta)$ is defined by (2.1.5). Let $a > 0$ and $p > 1$. Then the equation*

$$-\triangle u + \lambda u + a|u|^{p-1}u = 0, \tag{2.3.8}$$

*has at least three distinct solutions $0$, $u$ and $-u$, where $u \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$, $u \neq 0$ and $u \geq 0$.*

PROOF. It is clear that 0 is a solution. On the other hand, there is a solution that minimizes $J(u)$ on $H_0^1(\Omega) \cap L^{p+1}(\Omega)$ (see Remark 2.3.6), where

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2 + \frac{a}{p+1}\int_\Omega |u|^{p+1}.$$

We first claim that we can find a solution that minimizes $J$ and that is nonnegative. Indeed, remember that the minimizing solution is constructed by considering a minimizing sequence $(u_n)_{n\geq 0}$. Setting $v_n = |u_n|$, we have $J(v_n) = J(u_n)$, so that $(v_n)_{n\geq 0}$ is also a minimizing sequence, which produces a nonnegative solution. Since $-u$ is a solution whenever $u$ is a solution, it remains to show that the infimum of $J$ is negative, so that this solution is not identically 0. Since $\lambda < -\lambda_1$, there exists $\varphi \in H_0^1(\Omega)$ such that $\|\varphi\|_{L^2} = 1$ and $\|\nabla\varphi\|_{L^2}^2 \in (\lambda_1, -\lambda)$. By density, there exists

$\varphi \in C_c^\infty(\Omega)$ such that $\|\varphi\|_{L^2} = 1$ and $\|\nabla\varphi\|_{L^2}^2 \in (\lambda_1, -\lambda)$. Set $\mu = \|\nabla\varphi\|_{L^2}^2$. Given $t > 0$, we have

$$J(t\varphi) = \frac{t^2}{2}(\mu + \lambda) + t^{p+1}\frac{a}{p+1}\int_\Omega |\varphi|^{p+1}.$$

Since $\mu + \lambda < 0$, we have $J(t\varphi) < 0$ for $t$ small enough, thus $\inf J(u) < 0$. This completes the proof. □

REMARK 2.3.10. Note that if $a > 0$ and $\lambda \geq -\lambda_1$, the only solution of (2.3.8) is $u = 0$. Indeed, let $u$ be a solution, and multiply the equation (2.3.8) by $u$. It follows that

$$\int_\Omega |\nabla u|^2 + \lambda\int_\Omega u^2 + a\int_\Omega |u|^{p+1} = 0,$$

thus $u = 0$.

## 2.4. Constrained minimization

Consider the equation

$$\begin{cases} -\triangle u + \lambda u = a|u|^{p-1}u & \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \partial\Omega, \end{cases} \tag{2.4.1}$$

with $\lambda > -\lambda_1$ where $\lambda_1 = \lambda_1(-\Delta)$ is defined by (2.1.5), $a > 0$ and $1 < p < (N+2)/(N-2)$. A solution of (2.4.1) is a critical point of the functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2 - \frac{a}{p+1}\int_\Omega |u|^{p+1}, \tag{2.4.2}$$

for $u \in H_0^1(\Omega)$. It is clear that $u = 0$ is a trivial solution. If we look for a nontrivial solution, we cannot apply the global minimization technique of the preceding section, because $E$ is not bounded from below. (To see this, take $u = t\varphi$ with $\varphi \in C_c^\infty(\Omega)$, $\varphi \neq 0$, and let $t \to \infty$.)

In this section, we will solve the equation (2.4.1) by minimizing

$$\frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2,$$

on the set

$$\left\{ u \in H_0^1(\Omega) \cap L^{p+1}(\Omega);\ \int_\Omega |u|^{p+1} = 1 \right\},$$

i.e. we will solve a minimization problem with constraint. For that purpose, we need the notion of Lagrange multiplier.

THEOREM 2.4.1 (Lagrange multipliers). *Let $X$ be a Banach space, let $F, J \in C^1(X, \mathbb{R})$ and set*

$$M = \{v \in X;\ F(v) = 0\}.$$

*Let $S \subset M$, $S \neq \emptyset$, and suppose $x_0 \in S$ satisfies*

$$J(u_0) = \inf_{v \in S} J(v).$$

*If $F'(u_0) \neq 0$ and if $M \cap \{x \in X;\ \|x - u_0\|_X \leq \eta\} \subset S$ for some $\eta > 0$, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $J'(u_0) = \lambda F'(u_0)$.*

PROOF. Let $f = J'(u_0)$ and $g = F'(u_0)$. If $f = 0$, then $\lambda = 0$ is a Lagrange multiplier. Therefore, we may assume $f \neq 0$. Note that by assumption, we also have $g \neq 0$. We now proceed in two steps.

STEP 1. $g^{-1}(0) \subset f^{-1}(0)$. Set $X_0 = g^{-1}(0)$. Since $g \neq 0$, there exists $w \in X$ such that $g(w) = 1$. Consider now the mapping $\phi : X_0 \times \mathbb{R} \to \mathbb{R}$ defined by $\phi(v, t) = F(u_0 + v + tw)$. We have $\phi(0, 0) = 0$, $\partial_t\phi(0, 0) = g(w) = 1$, $\partial_v\phi(0, 0) = g_{|X_0} = 0$. By the implicit function theorem, there exist $\varepsilon > 0$ and a function

$t \in C^1(B_\varepsilon, \mathbb{R})$ such that $t(0) = 0$, $t'(0) = 0$, and $\phi(v, t(v)) = 0$ for all $v \in B_\varepsilon$. Here, $B_\varepsilon = \{v \in X_0;\ \|v\|_X < \varepsilon\}$. Therefore, $F(u_0 + v + t(v)w) = 0$ for all $v \in B_\varepsilon$, hence $u_0 + v + t(v)w \in M$ for all $v \in B_\varepsilon$. By taking $\varepsilon$ sufficiently small, we have $u_0 + v + t(v)w \in S$ for all $v \in B_\varepsilon$, thus in particular

$$J(u_0 + v + t(v)w) \geq J(u_0), \tag{2.4.3}$$

for all $v \in B_\varepsilon$. Let now $v \in g^{-1}(0)$, i.e. $(F'(u_0), v)_{X^\star, X} = 0$. We need to show that $v \in f^{-1}(0)$, i.e. $(J'(u_0), v)_{X^\star, X} = 0$. Let

$$\varphi(s) = J(u_0 + sv + t(sv)w) - J(u_0),$$

for $|s| < \varepsilon \|v\|_X^{-1}$. We have $\varphi(0) = 0$, and it follows from (2.4.3) that $\varphi(s) \geq 0$. Therefore, $\varphi'(0) = 0$. Since

$$\varphi'(0) = (J'(u_0), v + (t'(0), v)_{X^\star, X} w)_{X^\star, X} = (J'(u_0), v)_{X^\star, X},$$

the result follows.

STEP 2. Conclusion. Since $g \neq 0$, there exists $w \in X$ such that $g(w) = 1$. Set $\lambda = f(w)$. Given any $u \in X$, we have

$$g(u - g(u)w) = g(u) - g(u)g(w) = 0.$$

Therefore, $u - g(u)w \in g^{-1}(0)$, so that by Step 1, $u - g(u)w \in f^{-1}(0)$. It follows that $f(u - g(u)w) = 0$, i.e. $f(u) = g(u)f(w) = \lambda g(u)$. This means that $f = \lambda g$, i.e. $J'(u_0) = \lambda F'(u_0)$. □

We now give an application of Theorem 2.4.1 to the resolution of the equation (2.4.1).

THEOREM 2.4.2. *Let $\Omega$ be a bounded domain of $\mathbb{R}^N$. Suppose $\lambda > -\lambda_1$ where $\lambda_1 = \lambda_1(-\Delta)$ is defined by (2.1.5), $a > 0$ and $1 < p < (N+2)/(N-2)$ ($1 < p < \infty$ if $N = 1$ or $2$). Then there exists a solution $u \in H^1_0(\Omega)$, $u \geq 0$, $u \not\equiv 0$ of the equation (2.4.1).*

PROOF. Set

$$F(u) = \frac{1}{p+1} \int_\Omega |u|^{p+1} - 1,$$

and

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{\lambda}{2} \int_\Omega u^2.$$

It follows that $F, J \in C^1(H^1_0(\Omega), \mathbb{R})$ (Corollary 2.2.3). Let

$$M = S = \{u \in H^1_0(\Omega);\ F(u) = 0\}.$$

We have $F'(u) = |u|^{p-1}u \neq 0$ for all $u \in S$. We construct $u \in S$ such that

$$J(u) = \inf_{v \in S} J(v). \tag{2.4.4}$$

Since $J \geq 0$, we may consider a minimizing sequence $(u_n)_{n \geq 0} \subset S$, which is bounded in $H^1_0(\Omega)$ (by (2.1.8)). Set now $v_n = |u_n|$. It follows that $(v_n)_{n \geq 0} \subset S$ and is also a minimizing sequence. Therefore (Theorem 5.5.5), there exist a subsequence, which we still denote by $(v_n)_{n \geq 0}$, and $v \in H^1_0(\Omega)$ such that $v_n \to v$ in $L^{p+1}(\Omega)$ and $\|\nabla v\|_{L^2} \leq \liminf \|\nabla v_n\|_{L^2}$ as $n \to \infty$. It follows that $F(v) = 0$, i.e. $v \in S$ and $J(v) \leq \liminf_{n \to \infty} J(v_n)$. Thus $v$ satisfies (2.4.4). In addition, we have $v \geq 0$, and since $v \in S$, $v \not\equiv 0$. By Theorem 2.4.1, there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that $J'(v) = \mu F'(v)$, i.e.

$$-\triangle v + \lambda v = \mu |v|^{p-1} v. \tag{2.4.5}$$

Taking the $H^{-1} - H^1_0$ duality product of (2.4.5) with $v$, we obtain

$$2J(v) = \mu \int_\Omega |v|^{p+1}.$$

Since $v \neq 0$, we have $J(v) > 0$, and it follows that $\mu > 0$. Finally, set $u = (\mu/a)^{\frac{1}{p-1}} v$. It follows from (2.4.5) that $u$ satisfies (2.4.1). This completes the proof. □

REMARK 2.4.3. Here are some comments on Theorem 2.4.2.

(i) If, instead of the equation (2.4.1), we consider the equation $-\triangle u + \lambda u = a|u|^{p-1}u + h$, with $h \neq 0$, then the existence problem is considerably more difficult, and only partial results are known. See Struwe [43], Bahri and Berestycki [6], Bahri and Lions [7], Bahri [5].

(ii) If we replace the nonlinearity $|u|^{p-1}u$ by a nonhomogeneous one $g(u)$ with the same behavior (for example, $g(u) = |u|^{p-1}u + |u|^{q-1}u$), then the method we used to prove existence does not apply, because of the scaling used at the very end of the proof (which uses the homogeneity). In this case, what we obtain is the existence of $u \in H_0^1(\Omega)$ and $\mu > 0$ such that $-\triangle u + \lambda u = \mu g(u)$. In order to solve equations of the type (2.4.1) with nonhomogeneous nonlinearities, we will apply the mountain pass theorem in the next section.

(iii) The assumption $p < (N+2)/(N-2)$ may be essential or not, depending on the domain $\Omega$. See Section 2.7.

(iv) The assumption $\lambda > -\lambda_1$ is not essential for the existence of a nontrivial solution $u \in H_0^1(\Omega)$ of the equation (2.4.1). (See for example Kavian [28], Example 8.7 of Chapter 3.) However, it is necessary for the existence of a nontrivial solution $u \geq 0$. Indeed, suppose $u \geq 0$ is a solution of (2.4.1). Multiplying the equation by $\varphi_1$, a positive eigenvector corresponding to the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$ (see Section 3.2 below), we obtain

$$(\lambda_1 + \lambda) \int_\Omega u\varphi_1 = a \int_\Omega |u|^{p-1} u\varphi_1.$$

Since $\varphi_1 > 0$, the right-hand side is positive, so the left-hand side must be positive, which implies $\lambda > -\lambda_1$. Note that even when $\lambda \leq -\lambda_1$, we can apply the minimization technique of the proof of Theorem 2.4.2. It is clear that the minimization sequence $(v_n)_{n\geq 0}$ is bounded in $H_0^1(\Omega)$ (note that it is a priori bounded in $L^2(\Omega)$ since $v_n \in S$. Therefore, we obtain a solution $v \geq 0$, $v \neq 0$ of the equation (2.4.5). However, multiplying the equation (2.4.5) by $\varphi_1$, we see that $\mu = 0$ if $\lambda = -\lambda_1$ and $\mu < 0$ if $\lambda < -\lambda_1$. Therefore, the method applies, but it produces a solution of the equation (2.4.1) with $a \leq 0$.

Solutions of minimal energy $E$ (defined by (2.4.2)) may be important for some applications, because they tend to be "more stable", in some appropriate sense. However, we saw that the energy $E$ is not bounded from below, so a solution cannot minimize the energy on the whole space $H_0^1(\Omega)$. There is still an appropriate notion of solution of minimal energy, the *ground state*. A ground state is a nontrivial solution of (2.4.1) which *minimizes $E$ among all nontrivial solutions of* (2.4.1).

We will show below the existence of a ground state. We can use two arguments for that purpose:

– We can minimize $E(u)$ on the set
$$S = \Big\{ u \in H_0^1(\Omega);\ u \neq 0 \text{ and } \int_\Omega |\nabla u|^2 + \lambda \int_\Omega u^2 = a \int_\Omega |u|^{p+1} \Big\},$$
in order construct in one step a solution of (2.4.1) which is a ground state. In addition, we obtain the existence of a ground state $u \geq 0$.

– We can consider a minimizing sequence of nontrivial solutions of the equation (2.4.1) (which exists by Theorem 2.4.2) and show that some subsequence converges to a ground state.

We show below the existence of a ground state by using the first method. We will also prove a more general result in the following section (Theorem 2.5.8).

THEOREM 2.4.4. *Under the assumptions of Theorem 2.4.2, there exists a ground state $u \geq 0$ of the equation (2.4.1).*

PROOF. Let

$$F(u) = \int_\Omega |\nabla u|^2 + \lambda \int_\Omega u^2 - a \int_\Omega |u|^{p+1},$$

set

$$M = \{u \in H_0^1(\Omega);\, F(u) = 0\}, \qquad S = \{u \in M;\, u \neq 0\},$$

and consider $E$ defined by (2.4.2). Given any $v \in H_0^1(\Omega)$, $v \neq 0$, we see that $F(tu) = 0$ for some $t > 0$. Thus $S \neq \emptyset$. We proceed in four steps.

STEP 1. $(F'(v), v)_{H^{-1}, H_0^1} < 0$ and $(E'(v), v)_{H^{-1}, H_0^1} = 0$ for all $v \in S$. Indeed,

$$\begin{aligned}(F'(v), v)_{H^{-1}, H_0^1} &= (-2\triangle v + 2\lambda v - a(p+1)|v|^{p-1}v, v)_{H^{-1}, H_0^1} \\ &= 2\int_\Omega |\nabla v|^2 + 2\lambda \int_\Omega v^2 - a(p+1)\int_\Omega |v|^{p+1} \\ &= 2F(v) - a(p-1)\int_\Omega |v|^{p+1},\end{aligned}$$

from which we deduce the first property. Since $(E'(v), v)_{H^{-1}, H_0^1} = F(v)$, the second property follows.

STEP 2. There exists $\delta > 0$ such that $\|v\|_{L^{p+1}} \geq \delta$ for all $v \in S$. Indeed, since $F(v) = 0$ and $\lambda > -\lambda_1$, there exists a constant $C$ such that

$$\|v\|_{H^1}^2 \leq C\|v\|_{L^{p+1}}^{p+1},$$

for all $v \in S$. By Sobolev's inequality, we deduce that

$$\|v\|_{L^{p+1}}^2 \leq C\|v\|_{L^{p+1}}^{p+1},$$

from which the result follows.

STEP 3. There exists $u \in S$, $u \geq 0$, such that

$$E(u) = \inf_{v \in S} E(v) := m. \tag{2.4.6}$$

Indeed,

$$\begin{aligned}E(v) &= \frac{1}{2}F(v) + a\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_\Omega |v|^{p+1} \\ &= a\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_\Omega |v|^{p+1},\end{aligned} \tag{2.4.7}$$

for all $v \in S$, so that $m > 0$ by Step 2. Furthermore, it follows from (2.4.6) and (2.4.7) that

$$m = a\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\inf_{v \in S}\int_\Omega |v|^{p+1}. \tag{2.4.8}$$

Let $(u_n)_{n\geq 0} \subset S$ be a minimizing sequence for (2.4.6), hence for for (2.4.8). Replacing $u_n$ by $|u_n|$, we see that we may assume $u_n \geq 0$. Since $u_n \in S$ and $(u_n)_{n\geq 0}$ is bounded in $L^{p+1}(\Omega)$ (hence in $L^2(\Omega)$) by (2.4.8), we see that $(u_n)_{n\geq 0}$ is bounded in $H_0^1(\Omega)$. Therefore (Theorem 5.5.5), there exist a subsequence, which we still denote by $(u_n)_{n\geq 0}$, and $u \in H_0^1(\Omega)$, $u \geq 0$, such that $u_n \to u$ in $L^{p+1}(\Omega)$ and $\|\nabla u\|_{L^2} \leq \liminf \|\nabla u_n\|_{L^2}$ as $n \to \infty$. It follows that

$$a\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_\Omega |u|^{p+1} = m, \tag{2.4.9}$$

and $F(u) \le 0$. We deduce in particular that there exists $t \in (0,1]$ such that $F(tu) = 0$, i.e. $tu \in S$. Therefore,

$$m \le a\Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_\Omega |tu|^{p+1} = t^{p+1} a\Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_\Omega |u|^{p+1} = t^{p+1} m,$$

by (2.4.9). Since $m > 0$, this implies that $t = 1$. Therefore, $u \in S$ and thus $E(u) = m$ by (2.4.9) and (2.4.7).

STEP 4. Conclusion. Let $u$ be as in Step 3. By Step 1 we have $F'(u) \ne 0$; and so, we may apply Theorem 2.4.1. It follows that there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $E'(u) = \lambda F'(u)$. Since, by Step 1, $(E'(u), u)_{H^{-1},H^1_0} = 0$ and $(F'(v), v)_{H^{-1},H^1_0} \ne 0$, we must have $\lambda = 0$; and so $u$ is a solution of the equation (2.4.1). It remains to show that $E(v) \ge E(u)$ for all solutions $v \ne 0$ of (2.4.1). This is clear, since any solution $v$ of (2.4.1) satisfies $F(v) = 0$, i.e. $v \in S$, and $u$ minimizes $E$ on $S$. □

We now establish the existence of nontrivial solutions of (2.4.1) in some domains for supercritical nonlinearities, i.e. for $p \ge (N+2)/(N-2)$.

THEOREM 2.4.5. *Assume $N \ge 2$. Let $0 < R_0 < R_1 \le \infty$ and let $\Omega$ be the annulus $\{x \in \mathbb{R}^N;\ R_0 < |x| < R_1\}$. Suppose $\lambda > -\lambda_1$ where $\lambda_1 = \lambda_1(-\Delta)$ is defined by (2.1.5), $a > 0$ and $p > 1$. It follows that there exists a radially symmetric solution $u \in H^1_0(\Omega)$, $u \ge 0$, $u \ne 0$ of the equation (2.4.1).*

PROOF. Recall that if $w \in H^1(\mathbb{R}^N)$ is radially symmetric, then

$$|w(x)| \le \sqrt{2} |x|^{-\frac{N-1}{2}} \|w\|_{L^2} \|\nabla w\|_{L^2}, \tag{2.4.10}$$

for a.a. $x \in \mathbb{R}^N$ (see (5.6.13)). We denote by $W$ the subspace of $H^1_0(\Omega)$ of radially symmetric functions, so that $W$ is a closed subspace of $H^1_0(\Omega)$. Given $u \in W$, let

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \notin \Omega. \end{cases}$$

It follows that $\widetilde{u} \in H^1(\mathbb{R}^N)$. Since $\widetilde{u}$ is also radially symmetric, we may apply estimate (2.4.10) and we deduce that

$$|u(x)| \le \sqrt{2} |x|^{-\frac{N-1}{2}} \|u\|_{L^2} \|\nabla u\|_{L^2}, \tag{2.4.11}$$

for a.a. $x \in \Omega$. This implies in particular that $W \hookrightarrow L^\infty(\Omega)$, thus $W \hookrightarrow L^{p+1}(\Omega)$. We now argue as in Theorem 2.4.2. Set

$$F(u) = \int_\Omega |u|^{p+1} - 1,$$

and

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{\lambda}{2} \int_\Omega u^2.$$

It follows that $F, J \in C^1(W, \mathbb{R})$. (Apply Corollary 2.2.2 and the embedding $W \hookrightarrow L^{p+1}(\Omega)$.) Let

$$M = S = \{u \in W;\ F(u) = 0\}.$$

We have $F'(u) = |u|^{p-1}u \ne 0$ for all $u \in S$. We construct $v \in S$ such that

$$J(v) = \inf_{w \in S} J(w). \tag{2.4.12}$$

Since $J \ge 0$, we may consider a minimizing sequence $(u_n)_{n \ge 0} \subset S$, which is bounded in $W$ (by (2.1.8)). Set now $v_n = |u_n|$. It follows that $(v_n)_{n\ge 0} \subset S$ and is also a minimizing sequence. We now consider separately two cases.

CASE 1: $R_1 < \infty$. There exist a subsequence, which we still denote by $(v_n)_{n \ge 0}$, and $v \in W$ such that $v_n \to v$ in $L^2(\Omega)$ and $\|\nabla v\|_{L^2} \le \liminf \|\nabla v_n\|_{L^2}$

as $n \to \infty$ (Theorem 5.5.5). It follows that $J(v) \le \liminf J(v_n)$. Furthermore, since $W \hookrightarrow L^\infty(\Omega)$ and $v_n \to v$ in $L^2(\Omega)$, we deduce from Hölder's inequality that $v_n \to v$ in $L^{p+1}(\Omega)$. This implies that $F(v) = 0$; and so $v \in S$ satisfies (2.4.12)

CASE 2: $R_1 = \infty$. In this case, it follows from Remark 2.1.5 that $\lambda_1 = 0$, thus $\lambda > 0$. There exist a subsequence, which we still denote by $(v_n)_{n\ge 0}$, and $v \in W$ such that $v_n \to v$ in $L^r(\Omega)$ as $n \to \infty$ for all $2 < r < 2N/(N-2)$. Furthermore, $\|v\|_{L^2} \le \liminf \|v_n\|_{L^2}$ and $\|\nabla v\|_{L^2} \le \liminf \|\nabla v_n\|_{L^2}$ as $n \to \infty$. (The estimate (2.4.11) is essential for that compactness property, see Remark 5.6.5 and Lemma 5.5.3.) Since $\lambda > 0$, it follows that $J(v) \le \liminf J(v_n)$. Furthermore, since $W \hookrightarrow L^\infty(\Omega)$ and $v_n \to v$ in $L^2(\Omega)$, we deduce from Hölder's inequality that $v_n \to v$ in $L^{p+1}(\Omega)$ as $n \to \infty$. This implies that $F(v) = 0$; and so $v \in S$ satisfies (2.4.12)

We see that in both cases, $v$ satisfies (2.4.12). In addition, we have $v \ge 0$ and, since $v \in S$, $v \ne 0$. By Theorem 2.4.1, there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that $J'(v) = \mu F'(v)$, i.e.

$$-\triangle v + \lambda v = \mu |v|^{p-1} v. \tag{2.4.13}$$

Taking the $H^{-1} - H_0^1$ duality product of (2.4.13) with $v$, we obtain

$$2J(v) = \mu \int_\Omega |v|^{p+1}.$$

Since $v \ne 0$, we have $J(v) > 0$, and it follows that $\mu > 0$. Finally, set $u = (\mu/a)^{\frac{1}{p-1}} v$. It follows from (2.4.13) that $u$ satisfies (2.4.1). This completes the proof. □

REMARK 2.4.6. One can show that if $N \ge 2$, $\lambda > 0$, $a > 0$ and $1 < p < (N+2)/(N-2)$, then there exists a radially symmetric solution $u \in H^1(\mathbb{R}^N)$, $u \ge 0$, $u \ne 0$ of the equation (2.4.1). The proof is the same as the proof of Theorem 2.4.5 (use Theorem 5.6.3 for passing to the limit). Note that the upper bound on $p$ is essential by Pohožaev's identity (see Section 2.7 and in particular Lemma 2.7.1).

REMARK 2.4.7. Note that one cannot obtain ground states for the equations considered in Theorem 2.4.5 and Remark 2.4.6 by adaptating the argument that we used in the proof of Theorem 2.4.4 to the radial case. This would only prove the existence of a nontrivial solution that minimizes the energy among all nontrivial, radial solutions. We will obtain ground states by other methods (see Section 2.6).

## 2.5. The mountain pass theorem

In the preceding section, we established the existence of a nontrivial solution of the equation

$$\begin{cases} -\triangle u + \lambda u = f(u) & \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \partial\Omega, \end{cases} \tag{2.5.1}$$

in a bounded domain $\Omega$, with $\lambda > -\lambda_1$, and for homogeneous nonlinearities of the form $f(u) = a|u|^{p-1}u$ with $a > 0$ and $p < (N+2)/(N-2)$. The homogeneity of $f$ was essential for the method (constrained minimization). In this section, we will use the mountain pass theorem in order to establish existence of a nontrivial solution for nonhomogeneous nonlinearities.

We begin by establishing the mountain pass theorem, more precisely one of its many versions. We first introduce the Palais-Smale condition.

DEFINITION 2.5.1. Let $X$ be a Banach space and $J \in C^1(X, \mathbb{R})$. Given $c \in \mathbb{R}$, we say that $J$ satisfies the *Palais-Smale* condition at the level $c$ (in brief, $J$ satisfies $(\mathrm{PS})_c$) if the following holds. If there exists a sequence $(u_n)_{n\ge 0} \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ (in $X^\star$) as $n \to \infty$, then $c$ is a critical value (i.e.

there is $u \in X$ such that $J(u) = c$ and $J'(u) = 0$). We say that $J$ satisfies the Palais-Smale condition (in brief, $J$ satisfies (PS)) if $J$ satisfies $(\mathrm{PS})_c$ for all $c \in \mathbb{R}$.

We will see later examples of functionals that satisfy the Palais-Smale condition. We are now in a position to state the mountain pass theorem of Ambrosetti and Rabinowitz (see [**4**]).

THEOREM 2.5.2 (The mountain pass theorem). *Let $X$ be a Banach space, and let $J \in C^1(X, \mathbb{R})$. Suppose that:*

(i) *$J(0) = 0$;*
(ii) *there exist $\varepsilon, \gamma > 0$ such that $J(u) \geq \gamma$ for $\|u\| = \varepsilon$;*
(iii) *there exists $u_0 \in X$ such that $\|u_0\| > \varepsilon$ and $J(u) < \gamma$.*

*Set $\mathcal{A} = \{p \in C([0,1], X);\, p(0) = 0,\, p(1) = u_0\}$ and let*

$$c = \inf_{p \in \mathcal{A}} \max_{t \in [0,1]} J(p(t)) \geq \gamma.$$

*If $J$ satisfies $(\mathrm{PS})_c$, then $c$ is a critical value of $J$.*

COROLLARY 2.5.3. *Let $X$ be a Banach space and $J \in C^1(X, \mathbb{R})$. Suppose that:*

(i) *$J(0) = 0$;*
(ii) *there exist $\varepsilon, \gamma > 0$ such that $J(u) \geq \gamma$ for $\|u\| = \varepsilon$;*
(iii) *there exists $u_0 \in X$ such that $\|u_0\| > \varepsilon$ and $J(u) < \gamma$.*

*If $J$ satisfies (PS), then there exist $c \geq \gamma$ and $u \in X$ such that $J(u) = c$ and $J'(u) = 0$.*

Corollary 2.5.3 is an immediate consequence of Theorem 2.5.2. For the proof of Theorem 2.5.2, we follow the argument of Brezis and Nirenberg [**14**], which is especially simple and elegant. We will use the following two results.

LEMMA 2.5.4 (Ekeland's principle [**21**]). *Let $(\mathcal{A}, \mathrm{d})$ be a complete metric space and let $\psi \in C(\mathcal{A}, \mathbb{R})$ be bounded from below. If*

$$c = \inf_{p \in \mathcal{A}} \psi(p),$$

*then for every $\varepsilon > 0$, there exists $p_\varepsilon \in \mathcal{A}$ such that*

$$c \leq \psi(p_\varepsilon) \leq c + \varepsilon,$$

*and*

$$\psi(p) - \psi(p_\varepsilon) + \varepsilon \mathrm{d}(p, p_\varepsilon) \geq 0,$$

*for all $p \in \mathcal{A}$.*

PROOF. Fix $\varepsilon > 0$. Let $p_1 \in \mathcal{A}$ satisfy

$$c \leq \psi(p_1) \leq c + \varepsilon,$$

and set

$$E_1 = \{p \in \mathcal{A};\, \psi(p) - \psi(p_1) + \varepsilon \mathrm{d}(p, p_1) \leq 0\}.$$

It is clear that $p_1 \in E_1$, so that $E_1 \neq \emptyset$. Set

$$c_1 = \inf_{p \in E_1} \psi(p) \in [c, \psi(p_1)].$$

Fix $p_2 \in E_1$ such that

$$\psi(p_2) - c_1 \leq \frac{1}{2}(\psi(p_1) - c_1),$$

(observe that such a $p_2$ exists. Indeed, if $\psi(p_1) = c_1$, we take $p_2 = p_1$, and if $\psi(p_1) - c_1 > 0$, there exists a sequence $(p_{1,\ell})_{\ell \geq 0} \subset E_1$ such that $\psi(p_{1,\ell}) \to c_1$ as $\ell \to \infty$, so we take $p_2 = p_{1,\ell}$ for some $\ell$ large enough) and set

$$E_2 = \{p \in \mathcal{A};\, \psi(p) - \psi(p_2) + \varepsilon \mathrm{d}(p, p_2) \leq 0\}.$$

It is clear that $p_2 \in E_2$, so that $E_2 \neq \emptyset$. Set

$$c_2 = \inf_{p \in E_2} \psi(p) \in [c_1, \psi(p_2)].$$

We claim that $E_2 \subset E_1$. Indeed, if $p \in E_2$, then

$$\psi(p) - \psi(p_1) + \varepsilon \mathrm{d}(p, p_1) = [\psi(p) - \psi(p_2) + \varepsilon \mathrm{d}(p, p_2)] + [\psi(p_2) - \psi(p_1) + \varepsilon \mathrm{d}(p_2, p_1)] + \varepsilon [\mathrm{d}(p, p_1) - \mathrm{d}(p, p_2) - \mathrm{d}(p_2, p_1)] \leq 0.$$

Since $E_2 \subset E_1$, we see that $c_2 \geq c_1$. By induction, we construct a sequence $(p_n)_{n \geq 1} \subset \mathcal{A}$, a nonincreasing sequence $(E_n)_{n \geq 1}$ of nonempty, closed subsets of $\mathcal{A}$, and a nondecreasing sequence $(c_n)_{n \geq 1}$ of real numbers, $c \leq c_1 \leq \cdots \leq c + \varepsilon$. We have

$$\psi(p_{n+1}) - c_n \leq \frac{1}{2}(\psi(p_n) - c_n),$$

for all $n \geq 1$. Since the sequence $(c_n)_{n \geq 1}$ is nondecreasing, we deduce that

$$\psi(p_{n+1}) - c_{n+1} \leq \frac{1}{2}(\psi(p_n) - c_n);$$

and so,

$$\psi(p_{n+1}) - c_{n+1} \leq 2^{-n}(\psi(p_1) - c_1).$$

Furthermore, if $p \in E_{n+1}$, then by definition

$$\varepsilon \mathrm{d}(p, p_{n+1}) \leq \psi(p_{n+1}) - \psi(p) \leq \psi(p_{n+1}) - c_{n+1} \leq 2^{-n}(\psi(p_1) - c_1).$$

This means that the diameter of $E_n$ converges to 0 as $n \to \infty$. Since $\mathcal{A}$ is complete, it follows that $\cap_{n \geq 1} E_n$ is reduced to a point, which we denote by $p_\varepsilon$. Given now $p \in \mathcal{A}$, $p \neq p_\varepsilon$, we have $p \notin E_n$ for $n \geq n_0$; and so,

$$\psi(p) - \psi(p_n) + \varepsilon \mathrm{d}(p, p_n) > 0,$$

for $n \geq n_0$. Letting $n \to \infty$, we obtain

$$\psi(p) - \psi(p_\varepsilon) + \varepsilon \mathrm{d}(p, p_\varepsilon) \geq 0.$$

Since $p \neq p_\varepsilon$ is arbitrary, the result follows. □

LEMMA 2.5.5. *Let $X$ be a Banach space and let $f \in C([0,1], X^\star)$. For every $\varepsilon > 0$, there exists $v \in C([0,1], X)$ such that*

$$\|v(t)\|_X \leq 1,$$

*and*

$$(f(t), v(t))_{X^\star, X} \geq \|f(t)\|_{X^\star} - \varepsilon,$$

*for all $t \in [0,1]$.*

PROOF. Fix $\varepsilon > 0$. For every $t \in [0,1]$, there exists $x_t \in X$ such that

$$\|x_t\|_X < 1, \quad (f(t), x_t)_{X^\star, X} > \|f(t)\|_{X^\star} - \varepsilon.$$

By continuity, there exists $\delta(t) > 0$ such that

$$(f(s), x_t)_{X^\star, X} > \|f(s)\|_{X^\star} - \varepsilon,$$

for all $s \in [0,1]$ such that $|s - t| \leq \delta(t)$. In particular,

$$[0,1] \subset \bigcup_{t \in [0,1]} (t - \delta(t), t + \delta(t),$$

and we deduce by compactness of $[0,1]$ that there exist an integer $\ell \geq 1$ and $(t_j)_{1 \leq j \leq \ell} \subset [0,1]$ such that

$$[0,1] \subset \bigcup_{1 \leq j \leq \ell} I_j,$$

where $I_j = [0,1] \cap (t_j - \delta(t_j), t_j + \delta(t_j))$. If $\overline{I_j} = [0,1]$ for some $1 \le j \le \ell$, then we can take $v(t) \equiv x_{t_j}$. Otherwise, given any $1 \le j \le \ell$, we set

$$\rho_j(t) = \operatorname{dist}(t, K_j),$$

where $K_j = [0,1] \setminus I_j$, and

$$\rho(t) = \sum_{j=1}^{\ell} \rho_j(t),$$

for all $t \in [0,1]$. We observe that any $t \in [0,1]$ belongs to some $I_j$, so that $\rho_j(t) > 0$. In particular, $\rho(t) > 0$ for all $t \in [0,1]$. Finally, set

$$v(t) = \frac{1}{\rho(t)} \sum_{j=1}^{\ell} \rho_j(t) x_{t_j}.$$

We claim that $v$ satisfies the conclusions of the lemma. Indeed,

$$\|v(t)\|_X \le \frac{1}{\rho(t)} \sum_{j=1}^{\ell} \rho_j(t) \|x_{t_j}\|_X \le 1.$$

In addition, note that if $\rho_j(t) > 0$, then $t \in I_j$; and so,

$$\begin{aligned}(f(t), v(t))_{X^\star, X} &= \frac{1}{\rho(t)} \sum_{j=1}^{\ell} \rho_j(t) (f(t), x_{t_j})_{X^\star, X} \\ &\ge \frac{1}{\rho(t)} \sum_{j=1}^{\ell} \rho_j(t) (\|f(t)\|_{X^\star} - \varepsilon) \ge \|f(t)\|_{X^\star} - \varepsilon,\end{aligned}$$

which completes the proof. □

Proof of Theorem 2.5.2. Let $\mathrm{d}(p,q) = \|p - q\|_{C([0,1],X)}$ for all $p, q \in \mathcal{A}$, and set

$$\psi(p) = \max_{t \in [0,1]} J(p(t)),$$

for $p \in \mathcal{A}$. We note that $(\mathcal{A}, \mathrm{d})$ is a complete metric space and that $\psi \in C(\mathcal{A}, \mathbb{R})$. Therefore, we may apply Ekeland's principle and we see that for every $\varepsilon > 0$, there exists $p_\varepsilon \in \mathcal{A}$ such that

$$c \le \psi(p_\varepsilon) \le c + \varepsilon,$$

and

$$\psi(p) - \psi(p_\varepsilon) + \varepsilon \mathrm{d}(p, p_\varepsilon) \ge 0, \tag{2.5.2}$$

for all $p \in \mathcal{A}$. We claim that there exists $t_\varepsilon \in (0,1)$, such that

$$c \le J(p_\varepsilon(t_\varepsilon)) \le c + \varepsilon, \tag{2.5.3}$$

and

$$\|J'(p_\varepsilon(t_\varepsilon))\|_{X^\star} \le \varepsilon. \tag{2.5.4}$$

To see this, consider the set

$$B_\varepsilon = \{t \in [0,1];\ J(p_\varepsilon(t)) = \psi(p_\varepsilon)\}.$$

We need only show that there exists $t_\varepsilon \in B_\varepsilon$ such that

$$\|J'(p_\varepsilon(t_\varepsilon))\|_{X^\star} \le 2\varepsilon. \tag{2.5.5}$$

Applying Lemma 2.5.5 with $f(t) = J'(p_\varepsilon(t))$, we obtain a function $v_\varepsilon \in C([0,1], X)$ such that

$$\|v_\varepsilon(t)\|_X \le 1, \quad (J'(p_\varepsilon(t)), v_\varepsilon(t))_{X^\star, X} \ge \|J'(p_\varepsilon(t))\|_{X^\star} - \varepsilon, \tag{2.5.6}$$

for all $t \in [0,1]$. Since $B_\varepsilon \subset (0,1)$ (recall that $J(p_\varepsilon(0)), J(p_\varepsilon(1)) < c$) there exists a function $\alpha_\varepsilon \in C([0,1],\mathbb{R})$ such that $0 \le \alpha_\varepsilon \le 1$, $\alpha_\varepsilon(0) = \alpha_\varepsilon(1) = 0$ and $\alpha_\varepsilon \equiv 1$ on a neighborhood of $B_\varepsilon$. Given $n \ge 1$, we let

$$p(t) = p_\varepsilon(t) - \frac{1}{n}\alpha_\varepsilon(t)v_\varepsilon(t),$$

in (2.5.2), and we obtain

$$\psi(p_\varepsilon - n^{-1}\alpha_\varepsilon v_\varepsilon) - \psi(p_\varepsilon) + \frac{\varepsilon}{n} \ge 0. \tag{2.5.7}$$

We set

$$B_{\varepsilon,n} = \{t \in [0,1];\ J(p_\varepsilon(t) - n^{-1}\alpha_\varepsilon(t)v_\varepsilon(t)) = \psi(p_\varepsilon - n^{-1}\alpha_\varepsilon v_\varepsilon)\},$$

and we observe that $B_{\varepsilon,n} \neq \emptyset$ by definition of $\psi$. Consider a sequence $(t_{\varepsilon,n})_{n\ge1}$ with $t_{\varepsilon,n} \in B_{\varepsilon,n}$. There exist a subsequence, which we still denote by $(t_{\varepsilon,n})_{n\ge1}$ and $t_\varepsilon \in [0,1]$ such that $t_{\varepsilon,n} \to t_\varepsilon$ as $n \to \infty$. Note that, since $\psi$ is continuous, $\psi(p_\varepsilon - n^{-1}\alpha_\varepsilon v_\varepsilon) \to \psi(p_\varepsilon)$ as $n \to \infty$, so that

$$\begin{aligned} J(p_\varepsilon(t_\varepsilon)) &= \lim_{n\to\infty} J(p_\varepsilon(t_{\varepsilon,n}) - n^{-1}\alpha_\varepsilon(t_{\varepsilon,n})v_\varepsilon(t_{\varepsilon,n})) \\ &= \lim_{n\to\infty} \psi(p_\varepsilon - n^{-1}\alpha_\varepsilon v_\varepsilon) = \psi(p_\varepsilon). \end{aligned}$$

We deduce that $t_\varepsilon \in B_\varepsilon$. Note that for $n$ large enough, we have $\alpha_\varepsilon(t_{\varepsilon,n}) = 1$ (because $\alpha_\varepsilon \equiv 1$ on a neighborhood of $B_\varepsilon$), so that

$$\begin{aligned} &J(p_\varepsilon(t_{\varepsilon,n}) - n^{-1}v_\varepsilon(t_{\varepsilon,n})) - J(p_\varepsilon(t_{\varepsilon,n})) \\ &\quad = J(p_\varepsilon(t_{\varepsilon,n}) - n^{-1}\alpha_\varepsilon(t_{\varepsilon,n})v_\varepsilon(t_{\varepsilon,n})) - J(p_\varepsilon(t_{\varepsilon,n})) \\ &\qquad \ge J(p_\varepsilon(t_{\varepsilon,n}) - n^{-1}\alpha_\varepsilon(t_{\varepsilon,n})v_\varepsilon(t_{\varepsilon,n})) - \psi(p_\varepsilon), \end{aligned}$$

since $t_\varepsilon \in B_\varepsilon$. Therefore, since $t_{\varepsilon,n} \in B_{\varepsilon,n}$,

$$\begin{aligned} &J(p_\varepsilon(t_{\varepsilon,n}) - n^{-1}v_\varepsilon(t_{\varepsilon,n})) - J(p_\varepsilon(t_{\varepsilon,n})) \\ &\qquad \ge \psi(p_\varepsilon - n^{-1}\alpha_\varepsilon v_\varepsilon) - \psi(p_\varepsilon) \ge -\frac{\varepsilon}{n}, \end{aligned} \tag{2.5.8}$$

by (2.5.7). On the other hand,

$$\begin{aligned} &J(p_\varepsilon(t_{\varepsilon,n}) - n^{-1}v_\varepsilon(t_{\varepsilon,n})) - J(p_\varepsilon(t_{\varepsilon,n})) \\ &\qquad = \int_0^1 \frac{d}{ds} J(p_\varepsilon(t_{\varepsilon,n}) - sn^{-1}v_\varepsilon(t_{\varepsilon,n}))\, ds \\ &\qquad = -\frac{1}{n}\int_0^1 (J'(p_\varepsilon(t_{\varepsilon,n}) - sn^{-1}v_\varepsilon(t_{\varepsilon,n})), v_\varepsilon(t_{\varepsilon,n}))_{X^\star,X}\, ds; \end{aligned}$$

and so,

$$\begin{aligned} &J(p_\varepsilon(t_{\varepsilon,n}) - n^{-1}v_\varepsilon(t_{\varepsilon,n})) - J(p_\varepsilon(t_{\varepsilon,n})) + \frac{1}{n}(J'(p_\varepsilon(t_{\varepsilon,n})), v_\varepsilon(t_{\varepsilon,n}))_{X^\star,X} \\ &\quad = -\frac{1}{n}\int_0^1 \big(J'(p_\varepsilon(t_{\varepsilon,n}) - sn^{-1}v_\varepsilon(t_{\varepsilon,n})) - J'(p_\varepsilon(t_{\varepsilon,n})), v_\varepsilon(t_{\varepsilon,n})\big)_{X^\star,X}. \end{aligned}$$

Since $\|v_\varepsilon\|_X \le 1$ and since $J$ is $C^1$, it follows that the right-hand side of the above identity is $o(n^{-1})$. Therefore,

$$\begin{aligned} &J(p_\varepsilon(t_{\varepsilon,n}) - n^{-1}v_\varepsilon(t_{\varepsilon,n})) - J(p_\varepsilon(t_{\varepsilon,n})) \\ &\qquad = -\frac{1}{n}(J'(p_\varepsilon(t_{\varepsilon,n})), v_\varepsilon(t_{\varepsilon,n}))_{X^\star,X} + o(n^{-1}). \end{aligned} \tag{2.5.9}$$

We deduce from (2.5.8) and (2.5.9) that

$$(J'(p_\varepsilon(t_{\varepsilon,n})), v_\varepsilon(t_{\varepsilon,n}))_{X^\star,X} \le \varepsilon + o(1).$$

Using now (2.5.6), we see that

$$\|J'(p_\varepsilon(t_{\varepsilon,n}))\|_{X^\star} \le 2\varepsilon + o(1).$$

Letting $n \to \infty$, we obtain (2.5.5), which proves the claim (2.5.3)-(2.5.4). Finally, we let $\varepsilon = 1/n$ for $n \ge 1$, and we let $u_n = p_\varepsilon(t_\varepsilon))$. It follows from (2.5.3)-(2.5.4) that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$. The result now follows by applying the condition $(\mathrm{PS})_c$. □

We now give some applications of the mountain pass theorem.

THEOREM 2.5.6. *Assume $\Omega$ is a bounded domain of $\mathbb{R}^N$, and let $\lambda > -\lambda_1$ where $\lambda_1 = \lambda_1(-\Delta)$ is defined by (2.1.5). Let $f \in C(\mathbb{R},\mathbb{R})$ satisfy $f(0) = 0$, and suppose there exist $1 < p < (N+2)/(N-2)$ ($1 < p < \infty$ if $N = 1$ or $2$), $\nu < \lambda + \lambda_1$ and $\theta > 2$ such that*

$$|f(u)| \le C(1 + |u|^p) \quad \text{for all} \quad u \in \mathbb{R},$$
$$F(u) \le \frac{\nu}{2} u^2 \quad \text{for} \quad |u| \quad \text{small},$$
$$0 < \theta F(u) \le u f(u) \quad \text{for} \quad |u| \quad \text{large},$$

*where $F(u) = \displaystyle\int_0^u f(s)\,ds$. It follows that there exists a solution $u \in H^1_0(\Omega)$, $u \ne 0$, of the equation* (2.5.1).

PROOF. Set

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda}{2}\int_\Omega u^2 - \int_\Omega F(u). \tag{2.5.10}$$

We will show, by applying the mountain pass theorem, that there exists a critical point $u \in H^1_0(\Omega)$ of $J$ such that $J(u) > 0$ (and so, $u \ne 0$). We proceed in two steps.

STEP 1. $J$ satisfies (PS). Suppose $(u_n)_{n\ge 0} \subset H^1_0(\Omega)$ satisfies $J(u_n) \to c \in \mathbb{R}$ and $J'(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$. Since $J'(u_n) = -\triangle u_n + \lambda u_n - f(u_n)$, it follows that

$$(J'(u_n), u_n)_{H^{-1},H^1_0} = \int_\Omega |\nabla u_n|^2 + \lambda \int_\Omega u_n^2 - \int_\Omega u_n f(u_n);$$

and so,

$$2J(u_n) - (J'(u_n), u_n)_{H^{-1},H^1_0} = \int_\Omega (u_n f(u_n) - 2F(u_n)).$$

Note that $uf(u) \ge \theta F(u) - C$ for all $u \in \mathbb{R}$ and some constant $C$. Therefore,

$$2J(u_n) - (J'(u_n), u_n)_{H^{-1},H^1_0} \ge (\theta - 2)\int_\Omega F(u_n) - C|\Omega|.$$

We deduce that

$$(\theta - 2)\int_\Omega F(u_n) \le 2J(u_n) + \|J'(u_n)\|_{H^{-1}}\|u_n\|_{H^1} + C|\Omega|. \tag{2.5.11}$$

It follows that there exists a constant $C$ such that

$$\int_\Omega F(u_n) \le C + C\|u_n\|_{H^1}.$$

Therefore, by (2.1.8),

$$J(u_n) \ge \alpha\|u_n\|^2_{H^1} - C\|u_n\|_{H^1} - C,$$

with $\alpha$ given by (2.1.7); and so $(u_n)_{n\ge 0}$ is bounded in $H^1_0(\Omega)$. We deduce (Theorem 5.5.5) that there exist a subsequence, which we still denote by $(u_n)_{n\ge 0}$ and $u \in H^1_0(\Omega)$ such that $u_n \to u$ in $L^{p+1}(\Omega)$ as $n \to \infty$ and

$$\int_\Omega \nabla u_n \cdot \nabla\varphi \underset{n\to\infty}{\longrightarrow} \int_\Omega \nabla u \cdot \nabla\varphi,$$

for all $\varphi \in H^1_0(\Omega)$. Furthermore, we may also assume that there exists $h \in L^{p+1}(\Omega)$ such that $|u_n| \le h$ a.e. in $\Omega$. We deduce easily by dominated convergence and the growth assumption on $f$ that $f(u_n) \to f(u)$ in $L^{\frac{p+1}{p}}(\Omega)$, hence in $H^{-1}(\Omega)$, as $n \to \infty$. It follows that

$$\int_\Omega f(u_n)\varphi \underset{n\to\infty}{\longrightarrow} \int_\Omega f(u)\varphi,$$

for all $\varphi \in H^1_0(\Omega)$. Therefore,

$$(-\triangle u_n + \lambda u_n - f(u_n), \varphi)_{H^{-1},H^1_0} \underset{n\to\infty}{\longrightarrow} (-\triangle u + \lambda u - f(u), \varphi)_{H^{-1},H^1_0}.$$

Since $-\triangle u_n + \lambda u_n - f(u_n) = J'(u_n) \to 0$, it follows that $J'(u) = 0$. It now remains to show that $J(u) = c$. It follows from what precedes that $-\triangle u_n + \lambda u_n \to -\triangle u + \lambda u$ in $H^{-1}(\Omega)$. By Theorem 2.1.4, this implies that $u_n \to u$ in $H^1_0(\Omega)$; and so, $J(u) = \lim J(u_n) = c$ as $n \to \infty$.

STEP 2. Conclusion. We have $J(0) = 0$. In addition, there exists a constant $C$ such that $F(u) \le \dfrac{\nu}{2}u^2 + C|u|^{p+1}$ for all $u \in \mathbb{R}$; and so,

$$\int_\Omega F(u) \le \frac{\nu}{2}\int_\Omega u^2 + C\|u\|^{p+1}_{H^1}.$$

Therefore,

$$J(u) \ge \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\lambda - \nu}{2}\int_\Omega u^2 - C\|u\|^{p+1}_{H^1}.$$

Since $\lambda - \nu > -\lambda_1$, we deduce that there exists $\delta > 0$ such that

$$J(u) \ge \delta\|u\|^2_{H^1} - C\|u\|^{p+1}_{H^1}.$$

Therefore, setting $\varepsilon = (\delta/2C)^{\frac{1}{p-1}}$, we have $J(u) \ge \delta\varepsilon^2/2 > 0$ for $\|u\|_{H^1} = \varepsilon$. We claim that there exists $u \in H^1_0(\Omega)$ such that $\|u\|_{H^1} \ge \varepsilon$ and $J(u) < 0$. Indeed, for $s$ large, we have

$$\frac{f(s)}{F(s)} \ge \frac{\theta}{s};$$

and so, $F(s) \ge cs^\theta$ for $s$ large. Thus $F(u) \ge cs^\theta - C$ for all $s \ge 0$. Consider now $\psi \in C^\infty_{\rm c}(\Omega)$ such that $\psi \ge 0$ and $\psi \not\equiv 0$, and $t > 0$. We have

$$J(t\psi) \le \frac{t^2}{2}\Big(\int_\Omega |\nabla\psi|^2 + \lambda\int_\Omega \psi^2\Big) + C|\Omega| - ct^\theta\int_\Omega \psi^\theta. \tag{2.5.12}$$

Therefore, $J(t\psi) < 0$ for $t$ large enough, which proves the claim. Since $J$ satisfies (PS) by Step 1, it follows from what precedes that we may apply the mountain pass theorem, from which the result follows. □

REMARK 2.5.7. Here are some comments on Theorem 2.5.6.

(i) We see that Theorem 2.5.6 applies to more general nonlinearities than Theorem 2.4.2, because it does not require homogeneity. On the other hand, we do not know if the nontrivial solution that we construct is nonnegative.

(ii) Note that the assumption $\lambda > -\lambda_1$ is not essential in Theorem 2.5.6. However, the proof in the general case requires a slightly stronger assumption on $f$ (namely, we need $F \ge 0$) and a more general version of the mountain pass theorem (see for example Kavian [28], Example 8.7 of Chapter 3.).

(iii) Note that in Step 1 of the proof, we proved a slightly stronger property than (PS). We proved that if $(u_n)_{n\ge0} \subset H^1_0(\Omega)$ satisfies $J'(u_n) \to 0$ and $J(u_n) \to c \in \mathbb{R}$ as $n \to \infty$, then there exist a subsequence $(u_{n_k})_{k>0}$ and $u \in H^1_0(\Omega)$ such that $u_{n_k} \to u$ in $H^1_0(\Omega)$ as $k \to \infty$ (and so, $J(u) = c$ and $J'(u) = 0$). This property is sometimes used as the definition of the Palais-Smale condition.

We saw that the energy $J$ is not bounded from below (see (2.5.12)), so a solution cannot minimize the energy on the whole space $H_0^1(\Omega)$. However, there is still the notion of *ground state*, as in the preceding section. A ground state is a nontrivial solution of (2.5.1) which minimizes $J$ among all nontrivial solutions of (2.5.1). We now show the existence of a ground state, under slightly stronger assumptions on $f$ than in Theorem 2.5.6.

THEOREM 2.5.8. *Assume $\Omega$ is a bounded domain of $\mathbb{R}^N$, and let $\lambda > -\lambda_1$ where $\lambda_1 = \lambda_1(-\Delta)$ is defined by (2.1.5). Let $f \in C(\mathbb{R},\mathbb{R})$ satisfy $f(0) = 0$, and suppose there exist $1 < p < (N+2)/(N-2)$ ($1 < p < \infty$ if $N = 1$ or $2$), $\nu < \lambda + \lambda_1$ and $\theta > 2$ such that*

$$|f(u)| \le C(1+|u|^p) \quad \text{for all} \quad u \in \mathbb{R},$$
$$uf(u) \le \nu u^2 + C|u|^{p+1} \quad \text{for all} \quad u \in \mathbb{R},$$
$$0 < \theta F(u) \le uf(u) \quad \text{for} \quad |u| \quad \text{large},$$

*where $F(u) = \displaystyle\int_0^u f(s)\,ds$. It follows that there exists a ground state of the equation (2.5.1).*

PROOF. Since $f$ satisfies the assumptions of Theorem 2.5.6, there exists a nontrivial solution of (2.5.1). Let $\mathcal{E} \neq \emptyset$ be the set of nontrivial solutions of (2.5.1), and set

$$m = \inf_{v\in\mathcal{E}} J(v).$$

If $v \in \mathcal{E}$, then it follows from (2.5.11) that

$$\theta \int_\Omega F(v) \le 2J(v) + C|\Omega|.$$

Since $F$ is bounded from below, we deduce that $J(v)$ is bounded from below; and so, $m > -\infty$. Let now $(u_n)_{n\ge 0}$ be a minimizing sequence. Since $J'(u_n) = 0$ and $J(u_n) \to m \in \mathbb{R}$, it follows from Remark 2.5.7 (iii) that there exist a subsequence $(u_{n_k})_{k>0}$ and $u \in H_0^1(\Omega)$ such that $u_{n_k} \to u$ in $H_0^1(\Omega)$ as $k \to \infty$. In particular, $J(u) = m$ and $J'(u) = 0$. Therefore, it only remains to show that $u \neq 0$. Indeed, we have $(J'(u_n), u_n)_{H^{-1},H_0^1} = 0$, i.e.

$$\int_\Omega |\nabla u_n|^2 + \lambda \int_\Omega u_n^2 = \int_\Omega u_n f(u_n) \le \nu \int_\Omega u_n^2 + C\int_\Omega |u_n|^{p+1}$$
$$\le \nu \int_\Omega u_n^2 + C\|u_n\|_{H^1}^{p+1};$$

and so,

$$\int_\Omega |\nabla u_n|^2 + (\lambda - \nu)\int_\Omega u_n^2 \le C\|u_n\|_{H^1}^{p+1}.$$

Since $\lambda - \nu > -\lambda_1$, we deduce that

$$\|u_n\|_{H^1}^2 \le C\|u_n\|_{H^1}^{p+1};$$

and since $u_n \neq 0$, we conclude that $\|u_n\|_{H^1} \ge C^{-\frac{1}{p-1}}$. It follows that $\|u\|_{H^1} \ge C^{-\frac{1}{p-1}}$, so that $u \neq 0$. ☐

## 2.6. Specific methods in $\mathbb{R}^N$

In this section, we consider the case $\Omega = \mathbb{R}^N$, and we study the existence of nontrivial solutions, and in particular of ground states, of the equation (2.5.1).

Under appropriate assumptions on $f$, we already obtained an existence result in Section 1.3. However, the method we applied fails if we consider a nonlinearity $f$ that also depends on $x$ in a non-radial way. Also, it does not show the existence of a ground state. We also obtained an existence result in the homogeneous case by a global minimization technique (see Remark 2.4.6), but we observed that the method does not apply to show the existence of a ground state (see Remark 2.4.7). Note also that we cannot apply the mountain pass theorem, since the associated functional does not satisfy the Palais-Smale condition, due to the lack of compactness of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$.

We will show in this section, the existence of a ground state, by solving a relevant constrained minimization problem, as in Berestycki and Lions [8]. The resolution of that problem will be an opportunity for introducing two different tools which allow to circumvent the difficulties raised by the lack of compactness. Both tools apply to the situation we consider, but it may happen that for a given problem one tool applies but the other does not.

Throughout this section, we assume that

$$f \in C^1(\mathbb{R}), \quad f(0) = f'(0) = 0, \tag{2.6.1}$$

and that there exists $1 < p < (N+2)/(N-2)$ such that

$$|f(u)| \le C(1 + |u|^p), \tag{2.6.2}$$

for all $u \in \mathbb{R}$. We set

$$F(u) = \int_0^u f(s)\, ds, \tag{2.6.3}$$

and for some of the results we will assume that there exists $u_0 \in \mathbb{R}$ such that

$$F(u_0) - \frac{\lambda}{2} u_0^2 > 0, \tag{2.6.4}$$

where $\lambda > 0$ is a given number. Finally, we set

$$V(u) = \int_{\mathbb{R}^N} F(u) - \frac{\lambda}{2} \int_{\mathbb{R}^N} u^2, \tag{2.6.5}$$

for $u \in H^1(\mathbb{R}^N)$. If $f$ satisfies (2.6.1)-(2.6.2), then it follows from Corollary 2.2.2 that $V \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and that $V'(u) = f(u) - \lambda u$.

We recall that a *ground state* of (2.5.1) is a solution $u \neq 0$, $u \in H^1(\mathbb{R}^N)$ of (2.5.1), such that $J(u) \le J(v)$ for all solutions $v \neq 0$, $v \in H^1(\mathbb{R}^N)$ of (2.5.1). Here, $J$ is defined by (2.5.10). Our main result is the following.

THEOREM 2.6.1. *Let $N \ge 3$, $\lambda > 0$ and assume (2.6.1), (2.6.2) and (2.6.4). It follows that there exists a ground state $u$ of (2.5.1).*

The proof of Theorem 2.6.1 consists in two steps. First, one reduces the existence of a ground state to the resolution of a constrained minimization problem (Proposition 2.6.2 below). Next, one solves the minimization problem (Proposition 2.6.4 below). As a matter of fact, we give two different proofs of Proposition 2.6.4, one based on the concentration-compactness principle of P.-L. Lions, the other (under slightly more restrictive assumptions on $f$) based on symmetrization. Note that we assume $N \ge 3$ for simplicity. The case $N = 1$ is solved completely in Section 1.1, and the case $N = 2$ is more delicate (see for example Kavian [28], Théorème 5.1 p. 276).

We first reduce the existence of a ground state of (2.5.1) to the resolution of a constrained minimization problem.

PROPOSITION 2.6.2. *Suppose $N \ge 3$. Assume (2.6.1)-(2.6.2), and let $\lambda \in \mathbb{R}$. If there exists a solution $\widetilde{u} \in H^1(\mathbb{R}^N)$ of the minimization problem*

$$\begin{cases} V(\widetilde{u}) = 1, \\ \int_{\mathbb{R}^N} |\nabla \widetilde{u}|^2 = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla v|^2;\, v \in H^1(\mathbb{R}^N),\, V(v) = 1 \Big\} := m, \end{cases} \tag{2.6.6}$$

*then there exists a ground state $u$ of (2.5.1). More precisely, if $\widetilde{u}$ is a solution of (2.6.6), then $u$ defined by $u(x) = \widetilde{u}(\gamma x)$ with $\gamma = \sqrt{2N/(N-2)m}$ is a ground state of (2.5.1).*

REMARK 2.6.3. Note that if $\widetilde{u}$ satisfies (2.6.6), then in particular $V(\widetilde{u}) = 1$, so that $\widetilde{u} \neq 0$. It follows that $\|\nabla\widetilde{u}\|_{L^2} > 0$, so that $\gamma > 0$.

PROOF OF PROPOSITION 2.6.2. Let $\widetilde{u}$ be a solution of (2.6.6). It follows from Theorem 2.4.1 that there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that

$$-\triangle\widetilde{u} = \Lambda(f(\widetilde{u}) - \lambda\widetilde{u}). \tag{2.6.7}$$

Indeed, we need only verify that $V'(\widetilde{u}) \neq 0$ in order to apply Theorem 2.4.1. Note that $\widetilde{u} \in L^\infty(\mathbb{R}^N)$ by standard regularity results (see e.g. Corollary 4.4.3). Set $H(x) = F(x) - \lambda x^2/2$, so that $H'(x) = f(x) - \lambda x$, $V(u) = \int_{\mathbb{R}^N} H(u)$ and $V'(u) = H'(u)$. Since $u$ is bounded, we may modify the values of $H(x)$ for $x$ large without modifying $V(u)$ nor $V'(u)$. In particular, we may assume that $H'$ is bounded, so that $H(u) \in H^1(\mathbb{R}^N)$. Since $V(u) \neq 0$, we must have $H(u) \not\equiv 0$, so that $\nabla H(u) \not\equiv 0$ (see Proposition 5.1.11). Since $\nabla H(u) = H'(u)\nabla u$, we see that $H'(u) \not\equiv 0$, which proves the claim, hence (2.6.7) is established.

Since $\widetilde{u} \in L^\infty(\mathbb{R}^N)$, we deduce from Pohožaev's identity (see Lemma 2.7.1) that

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla\widetilde{u}|^2 = N\Lambda V(\widetilde{u}) = N\Lambda;$$

and so,

$$\Lambda = \frac{N-2}{2N}m.$$

Therefore, it follows from (2.6.7) that $u$ defined by $u(x) = \widetilde{u}(\gamma x)$ satisfies (2.5.1).

It remains to show that $u$ is a ground state of (2.5.1). We observe that

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \gamma^{2-N}\int_{\mathbb{R}^N} |\nabla\widetilde{u}|^2 = \gamma^{2-N}m = \Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}} m^{\frac{N}{2}}. \tag{2.6.8}$$

Suppose now that $v \neq 0$ is another solution of (2.5.1). It follows from Pohožaev's identity that

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 = NV(v); \tag{2.6.9}$$

and so

$$V(v) = \frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla v|^2.$$

Therefore, if we set $v(x) = \widetilde{v}(\mu x)$ with

$$\mu = \Big(\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla v|^2\Big)^{-\frac{1}{N}},$$

we see that $V(\widetilde{v}) = \mu^N V(v) = 1$, so that

$$\int_{\mathbb{R}^N} |\nabla\widetilde{v}|^2 \geq m.$$

It follows that

$$\int_{\mathbb{R}^N} |\nabla v|^2 = \mu^{2-N}\int_{\mathbb{R}^N} |\nabla\widetilde{v}|^2 \geq \mu^{2-N}m = \Big(\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla v|^2\Big)^{\frac{N-2}{N}} m;$$

and so,

$$\int_{\mathbb{R}^N} |\nabla v|^2 \geq \Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}} m^{\frac{N}{2}}.$$

Comparing with (2.6.8), we obtain

$$\int_{\mathbb{R}^N} |\nabla v|^2 \geq \int_{\mathbb{R}^N} |\nabla u|^2. \tag{2.6.10}$$

Finally, we observe that if $w$ is any solution of (2.5.1), then it follows from Pohožaev's identity (2.6.9) that

$$J(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 - V(w) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla w|^2.$$

In particular, (2.6.10) implies $J(v) \geq J(u)$, which completes the proof. □

We now study the existence for the problem (2.6.6).

PROPOSITION 2.6.4. *Suppose $N \geq 3$ and $\lambda > 0$, and assume (2.6.1)-(2.6.2) and (2.6.4). It follows that there exists a solution $\widetilde{u} \in H^1(\mathbb{R}^N)$ of the minimization problem (2.6.6).*

PROOF. We proceed in five steps.

STEP 1. $\{v \in H^1(\mathbb{R}^N);\ V(v) = 1\} \neq \emptyset$. Consider the function $v$ defined by

$$v(x) = \begin{cases} u_0 & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \geq 1, \end{cases}$$

where $u_0$ is as in (2.6.4). It follows from (2.6.4) that

$$\int_{\mathbb{R}^N} \Big( F(v) - \frac{\lambda}{2} v^2 \Big) > 0.$$

By convolution of $v$ with a smoothing sequence, we obtain a sequence $(v_n)_{n\geq 0} \subset C_c^\infty(\mathbb{R}^N)$ such that $v_n \to v$ in $L^{p+1}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ as $n \to \infty$. It follows that

$$\int_{\mathbb{R}^N} \Big( F(v_n) - \frac{\lambda}{2} v_n^2 \Big) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} \Big( F(v) - \frac{\lambda}{2} v^2 \Big).$$

Therefore, for $n$ large enough, we have $v_n \in H^1(\mathbb{R}^N)$ and $V(v_n) > 0$. Fixing such a $n$ and setting $w(x) = v_n(\mu x)$ with $\mu = V(v_n)^{\frac{1}{N}}$, we see that $w \in H^1(\mathbb{R}^N)$ and $V(w) = 1$.

STEP 2. If $(v_n)_{n\geq 0}$ is a minimizing sequence for the problem (2.6.6), then $(v_n)_{n\geq 0}$ is bounded in $H^1(\mathbb{R}^N)$ and is bounded from below in $L^2(\mathbb{R}^N)$ and in $L^{p+1}(\mathbb{R}^N)$. We first observe that by assumption, $\|\nabla v_n\|_{L^2}$ is bounded, and we estimate $\|v_n\|_{L^2}$. Since $V(v_n) = 1$, we have

$$\int_{\mathbb{R}^N} v_n^2 = \frac{2}{\lambda}\Big( \int_{\mathbb{R}^N} F(v_n) - 1 \Big) \leq \frac{2}{\lambda}\int_{\mathbb{R}^N} F(v_n). \tag{2.6.11}$$

On the other hand, it follows from (2.6.1)-(2.6.2) that for every $\varepsilon > 0$, there exists $C_\varepsilon$ such that

$$F(s) \leq \varepsilon s^2 + C_\varepsilon |s|^{p+1}. \tag{2.6.12}$$

Therefore, we deduce from (2.6.11) that

$$\int_{\mathbb{R}^N} v_n^2 \leq C \int_{\mathbb{R}^N} |v_n|^{p+1}. \tag{2.6.13}$$

Since $p+1 < 2N/(N-2)$, we deduce from Gagliardo-Nirenberg's inequality that

$$\int_{\mathbb{R}^N} |v_n|^{p+1} \leq C \|\nabla v_n\|_{L^2}^{\frac{N(p-1)}{2}} \|v_n\|_{L^2}^{\frac{(N+2)-p(N-2)}{2}},$$

so that (2.6.13) yields

$$\|v_n\|_{L^2}^{\frac{(N-2)(p-1)}{2}} \leq C \|\nabla v_n\|_{L^2}^{\frac{N(p-1)}{2}},$$

which establishes the upper estimate of $\|v_n\|_{L^2}$, hence of $\|v_n\|_{H^1}$. We now prove the lower estimate of $\|v_n\|_{L^2}$. We have

$$1 = -\frac{\lambda}{2}\int_{\mathbb{R}^N} v_n^2 + \int_{\mathbb{R}^N} F(v_n) \le C\int_{\mathbb{R}^N} |v_n|^{p+1},$$

by (2.6.12). Applying again Gagliardo-Nirenberg's inequality, we obtain

$$1 \le C\|\nabla v_n\|_{L^2}^{\frac{N(p-1)}{2}} \|v_n\|_{L^2}^{\frac{(N+2)-p(N-2)}{2}} \le C\|v_n\|_{L^2}^{\frac{(N+2)-p(N-2)}{2}},$$

which proves the desired estimate. Finally, the lower estimate of $\|v_n\|_{L^{p+1}}$ follows from (2.6.13).

STEP 3. $m > 0$. It is clear that $m \ge 0$. Suppose now by contradiction $m = 0$ and consider a minimizing sequence $(v_n)_{n\ge 0}$. Since $(v_n)_{n\ge 0}$ is bounded in $L^2(\mathbb{R}^N)$ and $m = 0$, we deduce from Gagliardo-Nirenberg's inequality that $\|u_n\|_{L^{p+1}} \to 0$ as $n \to \infty$, which contradicts the lower estimate of Step 2.

STEP 4. There exist a minimizing sequence $(u_n)_{n\ge 0}$ for the problem (2.6.6) and $u \in H^1(\mathbb{R}^N)$ such that $u_n \to u$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. Consider a minimizing sequence $(v_n)_{n\ge 0}$. It follows from Step 2 that $(v_n)_{n\ge 0}$ is bounded in $H^1(\mathbb{R}^N)$ and bounded from below in $L^2(\mathbb{R}^N)$. Therefore, we may assume without loss of generality that $\|v_n\|_{L^2} \to a > 0$ as $n \to \infty$. We now apply the concentration-compactness principle of P.-L. Lions, see Theorem 5.6.1. It follows that there exists a subsequence, which we still denote by $(v_n)_{n\ge 0}$ which satisfies one of the following properties.

(i) Compactness up to a translation: There exist $u \in H^1(\mathbb{R}^N)$ and a sequence $(y_n)_{n\ge 0} \subset \mathbb{R}^N$ such that $v_n(\cdot - y_n) \to u$ in $L^r(\mathbb{R}^N)$ as $k \to \infty$, for $2 \le r < 2N/(N-2)$.

(ii) Vanishing: $\|v_{n_k}\|_{L^r} \to 0$ as $k \to \infty$ for $2 < r < 2N/(N-2)$.

(iii) Dichotomy: There exist $0 < \mu < a$ and two bounded sequences $(w_n)_{n\ge 0}$ and $(z_n)_{n\ge 0}$ of $H^1(\mathbb{R}^N)$ with compact support such that, as $n \to \infty$,

$$\|w_n\|_{L^2}^2 \to \mu, \quad \|z_n\|_{L^2}^2 \to a - \mu, \tag{2.6.14}$$

$$\operatorname{dist}(\operatorname{supp} w_n, \operatorname{supp} z_n) \to \infty, \tag{2.6.15}$$

$$\|v_n - w_n - z_n\|_{L^r} \to 0 \text{ for } 2 \le r < 2N/(N-2), \tag{2.6.16}$$

$$\limsup \|\nabla w_n\|_{L^2}^2 + \|\nabla z_n\|_{L^2}^2 \le m. \tag{2.6.17}$$

We see that if (i) holds, then setting $u_n(\cdot) = v_n(\cdot - y_n)$, $(u_n)_{n\ge 0}$ is also a minimizing sequence which is relatively compact in $L^2(\mathbb{R}^N)$. Therefore, we need only rule out (ii) and (iii). Since $(v_n)_{n\ge 0}$ is bounded from below in $L^{p+1}(\mathbb{R}^N)$ by Step 2, it follows that (ii) cannot occur. We finally rule out (iii). It is convenient to introduce, for $\lambda > 0$,

$$m_\lambda = \inf\Big\{\int_{\mathbb{R}^N} |\nabla\varphi|^2;\, \varphi \in H^1(\mathbb{R}^N),\, V(\varphi) = \lambda\Big\}. \tag{2.6.18}$$

It follows easily from the scaling identity $V(\varphi(\mu\cdot)) = \mu^{-N}V(\varphi(\cdot))$ that

$$m_\lambda = \lambda^{\frac{N-2}{N}} m. \tag{2.6.19}$$

Since $m > 0$ (by Step 3), we deduce in particular that

$$m < m_\gamma + m_{1-\gamma}, \tag{2.6.20}$$

for $0 < \gamma < 1$. Assume now by contradiction that (iii) holds. Since $w_n$ and $z_n$ have disjoint support, we see that $V(w_n + z_n) = V(w_n) + V(z_n)$. Also, it follows from (2.6.16) that $V(v_n) - V(w_n + z_n) \to 0$; and so, $V(v_n) - V(w_n) - V(z_n) \to 0$.

Since $(w_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ are bounded in $H^1(\mathbb{R}^N)$ and $V(v_n) = 1$, we may assume without loss of generality that there exists $\gamma \in \mathbb{R}$ such that

$$V(w_n) \underset{n\to\infty}{\longrightarrow} 1-\gamma \quad \text{and} \quad V(z_n) \underset{n\to\infty}{\longrightarrow} \gamma. \tag{2.6.21}$$

We consider separately the different possible values of $\gamma$. If $\gamma < 0$, then in particular, $V(w_n) > 1$ for $n$ large. It follows in particular from (2.6.17) that $\|\nabla w_n\|_{L^2}^2 \leq m$. Since on the other hand $\|\nabla w_n\|_{L^2}^2 \geq m_{V(w_n)} > m$ by (2.6.19), we obtain a contradiction. If $\gamma > 1$, then we also get to a contradiction by considering the sequence $(z_n)_{n\geq 0}$. If $\gamma = 0$, then the argument of Step 2 shows that $(w_n)_{n\geq 0}$ is bounded from below in $L^{p+1}(\mathbb{R}^N)$. By Gagliardo-Nirenberg's inequality, this implies that $\|\nabla w_n\|_{L^2}$ is bounded from below. We now deduce from (2.6.17) that $\limsup \|\nabla z_n\|^2 < m$. Since $V(z_n) \to 1$, we may assume by scaling that $V(z_n) = 1$, and we get to a contradiction with the definition of $m$. If $\gamma = 1$, then we also get to a contradiction by inverting the roles of the sequences $(w_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$. Finally, if $\gamma \in (0, 1)$, then we easily deduce from (2.6.17) and (2.6.21) that $m_\gamma + m_{1-\gamma} \leq m$, which contradicts (2.6.20).

STEP 5. Conclusion. We apply Step 4. Since the minimizing sequence $(u_n)_{n\geq 0}$ is bounded in $H^1(\mathbb{R}^N)$ by Step 2, and since $u_n \to u$ in $L^2(\mathbb{R}^N)$, we deduce from Gagliardo-Nirenberg's inequality that $u_n \to u$ in $L^{p+1}(\mathbb{R}^N)$ as $n \to \infty$. In particular, $1 = V(u_n) \to V(u)$. In addition, it follows from (5.5.8) that $\|\nabla u\|_{L^2}^2 \leq \liminf \|\nabla u_n\|_{L^2}^2 = m$ as $n \to \infty$. Therefore, $u$ satisfies (2.6.6). ☐

We now give an alternative proof of Proposition 2.6.4, which is applicable when $f$ is odd. That alternative proof is based on the properties of the symmetric-decreasing rearrangement. (See for example Lieb and Loss [**31**]; Hardy, Littlewood and Pólya [**24**]. For a different approach, see Brock and Solynin [**16**].) Given a measurable set $E$ of $\mathbb{R}^N$, we denote by $E^*$ the ball of $\mathbb{R}^N$ centered at 0 and such that

$$|E^*| = |E|.$$

Accordingly, we set

$$1_E^* = 1_{E^*}.$$

Given now a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ such that $|\{|u| > t\}| < \infty$ for all $t > 0$, we set

$$f^*(x) = \int_0^\infty 1^*_{\{|u|>t\}}(x)\, dt, \tag{2.6.22}$$

for all $x \in \mathbb{R}^N$. It is not difficult to show that $u^*$ is nonnegative, radially symmetric and nonincreasing. Moreover, $u^*$ has the same distribution function as $u$, i.e.

$$|\{u^* \geq \lambda\}| = |\{|u| \geq \lambda\}|,$$

for all $\lambda > 0$. It follows from the above identity that if $\phi \in C(\mathbb{R})$ is continuous, nondecreasing, and $\phi(0) = 0$, then

$$\int_{\mathbb{R}^N} \phi(u^*) = \int_{\mathbb{R}^N} \phi(|u|).$$

(Integrate the function $\theta(\lambda, x) = 1_{\{\phi(u(x)) \geq \lambda\}}$ on $(0, \infty) \times \mathbb{R}^N$ and apply Fubini.) The assumption that $\phi$ is nondecreasing can be removed by writing $\phi = \phi_1 - \phi_2$, where $\phi_1$ and $\phi_2$ are nondecreasing. In particular, if $H \in C(\mathbb{R})$ is even, $H(0) = 0$, and if $H(u) \in L^1(\mathbb{R}^N)$, it follows that $H(u^*) \in L^1(\mathbb{R}^N)$ and that

$$\int_{\mathbb{R}^N} H(u^*) = \int_{\mathbb{R}^N} H(u). \tag{2.6.23}$$

One can show that if $\nabla u \in L^p(\mathbb{R}^N)$ for some $1 \le p < \infty$, then $\nabla u^* \in L^p(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |\nabla u^*|^p \le \int_{\mathbb{R}^N} |\nabla u|^p. \tag{2.6.24}$$

This result, however, is more delicate. See Lieb [30] for a relatively simple proof in the case $p = 2$. See Brock and Solynin [16] for a really simple proof in the general case, via polarization.

Alternative proof of Proposition 2.6.4 when $f$ is odd. We only give an alternative proof of Steps 4 and 5. Note that, since $f$ is odd, $F$ is even. Consider a minimizing sequence $(v_n)_{n\ge 0}$ of the problem (2.6.6), and let $u_n = v_n^*$. It follows from (2.6.23) that $V(u_n) = 1$, and it follows from (2.6.24) that $(u_n)_{n\ge 0}$ is also a minimizing sequence. Since $u_n$ is spherically symmetric, it follows from Theorem 5.6.3 that there exist a subsequence, which we still denote by $(u_n)_{n\ge 0}$, and $u \in H^1(\mathbb{R}^N)$ such that $u_n \to u$ in $L^r(\mathbb{R}^N)$ as $n \to \infty$, for every $2 < r < 2N/(N-2)$. Note that for every $\varepsilon > 0$, there exists $C_\varepsilon$ such that $|F(x) - F(y)| \le \varepsilon|x-y| + C_\varepsilon(|x|^p + |y|^p)|x-y|$. Therefore, we see that

$$\int_{\mathbb{R}^N} F(u_n) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} F(u).$$

Since also

$$\int_{\mathbb{R}^N} u^2 \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^2,$$

and

$$\int_{\mathbb{R}^N} |\nabla u|^2 \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2,$$

by (5.5.6) and (5.5.8), we see that $V(u) \ge 1$ and $\|\nabla u\|_{L^2}^2 \le m$. It now remains to show that $V(u) = 1$. Suppose by contradiction that $V(u) > 1$, and set $v(x) = u(\mu x)$ with $\mu = (V(u))^{\frac{1}{N}} > 1$. It follows that $V(v) = 1$ and that $\|\nabla v\|_{L^2}^2 = \mu^{-(N-2)}\|\nabla u\|_{L^2}^2 \le \mu^{-(N-2)} m < m$, which contradicts the definition of $m$. This completes the proof. □

## 2.7. Study of a model case

In this section, we apply the results obtained in Chapters 1 and 2 to a model case, and we discuss the optimality. For the study of optimality, the following results, known as Pohožaev's identity, will be useful. The first one concerns the case $\Omega = \mathbb{R}^N$.

Lemma 2.7.1 (Pohožaev's identity). *Let $g \in C(\mathbb{R})$ and set $G(u) = \int_0^u g(s)\,ds$ for all $u \in \mathbb{R}$. If $u \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ satisfies*

$$-\triangle u = g(u), \tag{2.7.1}$$

*in $\mathcal{D}'(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \Big\{ \frac{N-2}{2}|\nabla u|^2 - NG(u) \Big\} = 0, \tag{2.7.2}$$

*provided $G(u) \in L^1(\mathbb{R}^N)$ and $\nabla u \in L^2(\mathbb{R}^N)$.*

Proof. We use the argument of Berestycki and Lions [8], proof of Proposition 1, p. 320. It follows from local regularity (see Theorem 4.4.5) that $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ for all $1 < p < \infty$ and all $0 \le \alpha < 1$. A long but straightforward calculation shows that, given any $x_0 \in \mathbb{R}^N$,

$$0 = [-\triangle u - g(u)][(x - x_0)\cdot \nabla u] = -\Big\{ \frac{N-2}{2}|\nabla u|^2 - NG(u) \Big\} + \nabla \cdot \Big\{ \Big( \frac{1}{2}|\nabla u|^2 - G(u) \Big)(x - x_0) - ((x - x_0)\cdot\nabla u)\nabla u \Big\}, \tag{2.7.3}$$

a.e. in $\mathbb{R}^N$. Choosing $x_0 = 0$ and integrating (2.7.3) on $B_R$ (the ball of $\mathbb{R}^N$ of center 0 and radius $R$), we obtain

$$\int_{B_R} \Big\{ \frac{N-2}{2} |\nabla u|^2 - NG(u) \Big\} = \int_{\partial B_R} \Big\{ \Big( \frac{1}{2} |\nabla u|^2 - G(u) \Big) x - (x \cdot \nabla u) \nabla u \Big\} \cdot \vec{n}. \quad (2.7.4)$$

Since $\nabla u \in L^2(\mathbb{R}^N)$ and $G(u) \in L^1(\mathbb{R}^N)$, we see that

$$\int_{B_R} \Big\{ \frac{N-2}{2} |\nabla u|^2 - NG(u) \Big\} \underset{R \to \infty}{\longrightarrow} \int_{\mathbb{R}^N} \Big\{ \frac{N-2}{2} |\nabla u|^2 - NG(u) \Big\}. \quad (2.7.5)$$

Moreover,

$$\int_0^\infty \int_{\partial B_R} (|\nabla u|^2 + |G(u)|) \, d\sigma \, dR = \int_{\mathbb{R}^N} (|\nabla u|^2 + |G(u)|) < \infty,$$

so that there exists a sequence $R_n \to \infty$ such that

$$R_n \int_{\partial B_{R_n}} (|\nabla u|^2 + |G(u)|) \underset{n \to \infty}{\longrightarrow} 0. \quad (2.7.6)$$

We finally let $R = R_n$ in (2.7.4) and let $n \to \infty$. It follows from (2.7.6) that the right-hand converges to 0 as $n \to \infty$. Since the limit of the left-hand side is given by (2.7.5), we obtain (2.7.2) in the limit. □

For the case of a general domain, we have the following result.

LEMMA 2.7.2 (Pohožaev's identity). *Let $g$ and $G$ be as above. Let $\Omega$ be an open domain of $\mathbb{R}^N$ with boundary of class $C^1$. If $u \in H^2(\Omega) \cap H^1_0(\Omega)$ is such that $g(u) \in H^{-1}(\mathbb{R}^N)$ and $G(u) \in L^1(\mathbb{R}^N)$ and satisfies (2.7.1) in $\mathcal{D}'(\Omega)$, then for any $x_0 \in \mathbb{R}^N$,*

$$\int_\Omega \Big\{ \frac{N-2}{2} |\nabla u|^2 - NG(u) \Big\} + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x - x_0) \cdot \vec{n} = 0, \quad (2.7.7)$$

*where $\vec{n}$ denotes the outward unit normal.*

PROOF. We give a formal argument, and we refer to Kavian [28], p. 253 for its justification. Integrating (2.7.2) on $\Omega$, and using the property $G(u) = 0$ on $\partial\Omega$, we obtain

$$0 = -\int_\Omega \Big\{ \frac{N-2}{2} |\nabla u|^2 - NG(u) \Big\} + \int_{\partial\Omega} \Big\{ \frac{1}{2} |\nabla u|^2 (x - x_0) - ((x - x_0) \cdot \nabla u) \nabla u \Big\} \cdot \vec{n}.$$

Since $u = 0$ on $\partial\Omega$, we see that $\nabla u \parallel \vec{n}$, so that $\nabla u = (\nabla u \cdot \vec{n}) \vec{n}$. Therefore,

$$((x - x_0) \cdot \nabla u) \nabla u = (\nabla u \cdot \vec{n})^2 ((x - x_0) \cdot \vec{n}) \vec{n} = |\nabla u|^2 ((x - x_0) \cdot \vec{n}) \vec{n},$$

and we obtain (2.7.7). □

We now consider the equation

$$\begin{cases} -\triangle u = -\lambda u + \mu |u|^{p-1} u & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \quad (2.7.8)$$

where $p > 1$ and $\lambda, \mu \in \mathbb{R}$.

We will consider three examples of domains $\Omega$.

Case 1: $\Omega = \mathbb{R}^N$. In this case, (2.7.2) has the form

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 = -\frac{N\lambda}{2}\int_{\mathbb{R}^N}u^2 + \frac{N\mu}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}, \tag{2.7.9}$$

provided $u \in H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N)$. In addition, multiplying the equation by $u$, we obtain

$$\int_{\mathbb{R}^N}|\nabla u|^2 = -\lambda\int_{\mathbb{R}^N}u^2 + \mu\int_{\mathbb{R}^N}|u|^{p+1}, \tag{2.7.10}$$

under the same assumptions on $u$.

- Suppose first $\mu < 0$.
  - If $\lambda \geq 0$, then it follows immediately from (2.7.10) that the unique solution of (2.7.8) in $H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N)$ is $u \equiv 0$.
  - If $\lambda < 0$, then the unique solution of (2.7.8) in the space $H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N)$ is $u \equiv 0$. This follows from the delicate result of Kato [27] (see also Agmon [2]). See also Remark 1.3.9 (vii) for the radial case and Remark 1.1.5 (ii) for the case $N = 1$.
- Suppose now $\mu > 0$.
  - If $\lambda = 0$, then it follows from (2.7.9)-(2.7.10) that

$$\Big(\frac{N-2}{2} - \frac{N}{p+1}\Big)\int_{\mathbb{R}^N}|\nabla u|^2 = 0.$$

    Therefore, if $N = 1, 2$, or if $N \geq 3$ and $p \neq (N+2)/(N-2)$, then the unique solution of (2.7.8) in $H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N)$ is $u \equiv 0$. If $N \geq 5$ and $p = (N+2)/(N-2)$, then there is a (radially symmetric, positive) solution $u \neq 0$ in $H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N)$ (see Remark 1.3.9 (iv)).
  - If $\lambda < 0$, then the unique solution of (2.7.8) in the space $H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N)$ is $u \equiv 0$. (See above.)
  - Suppose $\lambda > 0$. If $N = 1, 2$ or if $N \geq 3$ and $p < (N+2)/(N-2)$, then there is a solution $u \in H^1(\mathbb{R}^N)$, $u \neq 0$ of (2.7.8). Moreover, there is a positive, spherically symmetric solution. (See for example Theorem 1.3.1 for the case $N \geq 2$ and Remark 1.1.5 (ii) for the case $N = 1$.) If $N \geq 3$ and $p \geq (N+2)/(N-2)$, then it follows from (2.7.9)-(2.7.10) that

$$\Big(\frac{N-2}{2} - \frac{N}{p+1}\Big)\int_{\mathbb{R}^N}|\nabla u|^2 + \lambda\Big(\frac{N}{2} - \frac{N}{p+1}\Big)\int_{\mathbb{R}^N}u^2 = 0,$$

    so that the unique solution of (2.7.8) in $H^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N)$ is $u \equiv 0$.

Case 2: $\Omega = \{x \in \mathbb{R}^N;\ |x| < 1\}$. In this case, (2.7.7) has the form (taking $x_0 = 0$)

$$\frac{N-2}{2}\int_\Omega|\nabla u|^2 = -\frac{N\lambda}{2}\int_\Omega u^2 + \frac{N\mu}{p+1}\int_\Omega|u|^{p+1} - \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2, \tag{2.7.11}$$

provided $u \in H^2(\Omega)\cap L^{p+1}(\Omega)$. In addition, multiplying the equation by $u$, we obtain (2.7.10) under the same assumptions on $u$. Let $\lambda_1 = \lambda_1(-\Delta)$ be defined by (2.1.5).

- Suppose first $\mu < 0$.
  - If $\lambda \geq -\lambda_1$, then the unique solution of (2.7.8) in $H^1(\Omega)\cap L^{p+1}(\Omega)$ is $u \equiv 0$. (See Remark 2.3.10.)
  - If $\lambda < -\lambda_1$, then there is a solution $u \not\equiv 0$, $u \in H^1(\Omega)\cap L^{p+1}(\Omega)$ of (2.7.8). Moreover, there is a positive, radially symmetric solution. (See Theorem 2.3.9 for the existence of a positive solution and Theorem 4.5.1 for the symmetry.)
- Suppose now $\mu > 0$.

– Suppose $N \le 2$ or $N \ge 3$ and $p < (N+2)/(N-2)$. If $\lambda > -\lambda_1$, then there is a solution $u \not\equiv 0$, $u \in H^1(\Omega) \cap L^{p+1}(\Omega)$ of (2.7.8). Moreover, there is a positive, radially symmetric solution. (See Theorem 2.4.2 for the existence of a positive solution and Theorem 4.5.1 for the symmetry.) If $\lambda \le -\lambda_1$, then there is a solution $u \not\equiv 0$, $u \in H^1(\Omega) \cap L^{p+1}(\Omega)$ of (2.7.8). (See Remark 2.4.3 (iii).) These solutions are smooth, i.e. $u \in C_0(\Omega)$, see Remark 4.4.4.
– Suppose $N \ge 3$ and $p = (N+2)/(N-2)$. If $\lambda \ge 0$, then the unique solution of (2.7.8) in $H^2(\Omega)$ is $u \equiv 0$. Indeed, it follows from (2.7.10)-(2.7.11) that

$$\frac{N\lambda(p-1)}{2(p+1)} \int_\Omega u^2 = -\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2.$$

The conclusion follows if $\lambda > 0$. The case $\lambda = 0$ is more delicate: one observes that $\nabla u = 0$ a.e. on $\partial\Omega$, and this implies $u \equiv 0$ (see Pohožaev [40]). In fact, one can show that a solution in $H^1(\Omega)$ belongs to $H^2(\Omega)$, so that the unique solution in $H^1(\Omega)$ is $u \equiv 0$. If $N \ge 4$ and $-\lambda_1 < \lambda < 0$, there is a positive solution. This is a difficult result of Brezis and Nirenberg [15]. If $N = 3$ and $-\lambda_1 < \lambda < -\lambda_1/4$, then there is a positive solution (see [15]). If $N = 3$ and $\lambda \le -\lambda_1$, there is a solution $u \ne 0$ (see Comte [20]). If $N = 4$, $\lambda \le -\lambda_1$ and $\lambda \ne -\lambda_k$ for all $k \ge 1$, then there is a solution $u \ne 0$ (see Cerami, Solimini and Struwe [19]). If $N \ge 5$ and $\lambda \le -\lambda_1$, then there is a solution $u \ne 0$ (see [19]). The case $N = 4$ and $\lambda = -\lambda_k$ seems to be open. The case $N = 3$ and $\lambda \in [-\lambda_1/4, 0)$ is a very challenging open problem, which probably requires some new ideas. Note that in this last case, it is known that there is no nontrivial radial solution, and in particular, there is no positive solution (see Brezis and Nirenberg [15]).
– Suppose $N \ge 3$ and $p > (N+2)/(N-2)$. If

$$\lambda \ge -\lambda_1 \Big(1 - \frac{2(p+1)}{N(p-1)}\Big),$$

then the unique solution of (2.7.8) in $H^2(\Omega) \cap L^{p+1}(\Omega)$ is $u \equiv 0$. Indeed, it follows from (2.7.10)-(2.7.11) that

$$\Big(\frac{N-2}{2} - \frac{N}{p+1}\Big) \int_\Omega |\nabla u|^2 + \frac{N\lambda(p-1)}{2(p+1)} \int_\Omega u^2 = -\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2;$$

and so,

$$\Big[\Big(\frac{N-2}{2} - \frac{N}{p+1}\Big)\lambda_1 + \frac{N\lambda(p-1)}{2(p+1)}\Big] \int_\Omega u^2 \le -\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2.$$

The conclusion follows as above. If $N = 3$, then there exists $\lambda^* \in (0, \lambda_1)$ such that for $\lambda \in (-\lambda_1, -\lambda^*)$ there is a positive, radial solution (see Budd and Nurbury [17]). Also, one can show that there is always a bifurcation branch starting from $\lambda_k$, for every $k \ge 1$. The other cases seem to be open.

CASE 3: $\Omega = \{x \in \mathbb{R}^N;\, 1 < |x| < 2\}$. Let $\lambda_1 = \lambda_1(-\Delta)$ be defined by (2.1.5).

– Suppose first $\mu < 0$.
  – If $\lambda \ge -\lambda_1$, then the unique solution of (2.7.8) in $H^1(\Omega) \cap L^{p+1}(\Omega)$ is $u \equiv 0$. (See Remark 2.3.10.)
  – If $\lambda < -\lambda_1$, then there is a solution $u \not\equiv 0$, $u \in H^1(\Omega) \cap L^{p+1}(\Omega)$ of (2.7.8). Moreover, there is a positive, radially symmetric solution (See Theorem 2.3.9. In fact, Theorem 2.3.9 produces a positive solution, but one can construct a radially symmetric solution by minimizing on radially symmetric functions.) This solution is smooth, i.e. $u \in C_0(\Omega)$, see Remark 4.4.4.
– Suppose now $\mu > 0$.

- If $\lambda > -\lambda_1$, then there is a solution $u \not\equiv 0$, $u \in H^1(\Omega) \cap L^{p+1}(\Omega)$ of (2.7.8). Moreover, there is a positive, radially symmetric solution. (See Theorem 2.4.5 for the case $N \geq 2$ and Remark 1.2.6 for the case $N = 1$.) This solution is smooth, i.e. $u \in C_0(\Omega)$, see Remark 4.4.4.
- If $\lambda \leq -\lambda_1$, then there is a spherically symmetric solution $u \not\equiv 0$, $u \in H^1(\Omega) \cap L^{p+1}(\Omega)$ of (2.7.8). (See Kavian [**28**], Exemple 8.7 p. 173. Note that the assumption $p < (N+2)/(N-2)$ in [**28**] is not essential. It is used only for the verification of the Palais-Smale condition, which holds in the present case because of the embedding $W \hookrightarrow L^\infty(\Omega)$, see Theorem 2.4.5.) This solution is smooth, i.e. $u \in C_0(\Omega)$, see Remark 4.4.4.

CHAPTER 3

# Methods of super- and subsolutions

## 3.1. The maximum principles

Let $\Omega$ be an open subset of $\mathbb{R}^N$. We recall that a distribution $f \in H^{-1}(\Omega)$ is nonnegative (respectively, nonpositive), i.e. $f \geq 0$ (respectively, $f \leq 0$) if and only if $(f, \varphi)_{H^{-1}, H_0^1} \geq 0$ (respectively, $(f, \varphi)_{H^{-1}, H_0^1} \leq 0$), for all $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$. By density, this is equivalent to saying that $f \geq 0$ in $\mathcal{D}'(\Omega)$, i.e. that $(f, \varphi)_{\mathcal{D}', \mathcal{D}} \geq 0$ for all $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$. In particular, if $f \in L^1_{\mathrm{loc}}(\Omega) \cap H^{-1}(\Omega)$, then $f \geq 0$ in $H^{-1}(\Omega)$ if and only if $f \geq 0$ a.e. in $\Omega$. Therefore, the above definition is consistent with the usual one for functions.

We now can state the following weak form of the maximum principle.

THEOREM 3.1.1. *Consider a function $a \in L^\infty(\Omega)$, let $\lambda_1(-\triangle + a)$ be defined by (2.1.5) and let $\lambda > -\lambda_1(-\triangle + a)$. Suppose $u \in H^1(\Omega)$ satisfies $-\triangle u + au + \lambda u \geq 0$ (respectively, $\leq 0$) in $H^{-1}(\Omega)$. If $u^- \in H_0^1(\Omega)$(respectively, $u^+ \in H_0^1(\Omega)$), then $u \geq 0$ (respectively, $u \leq 0$) almost everywhere in $\Omega$.*

PROOF. We prove the first part of the result, the second follows by changing $u$ to $-u$. Since $u^- \geq 0$, we see that

$$(-\Delta u + au + \lambda u, -u^-)_{H^{-1}, H_0^1} \leq 0,$$

which we rewrite, using formula (5.1.5), as

$$\int_\Omega [\nabla u \cdot \nabla(-u^-) + au(-u^-) + \lambda u(-u^-)] \leq 0.$$

This means (see Remark 5.3.4) that

$$\int_\Omega [|\nabla(u^-)|^2 + a|u^-|^2 + \lambda |u^-|^2] \leq 0.$$

Since $\lambda > -\lambda_1(-\Delta + a)$, we deduce from (2.1.6) that $u^- = 0$, i.e. $u \geq 0$. □

We now study the strong maximum principle. Our first result in this direction is the following.

THEOREM 3.1.2. *Suppose $\Omega \subset \mathbb{R}^N$ is a connected, open set. Let $\lambda_1(-\triangle)$ be defined by (2.1.5) and let $\lambda > -\lambda_1(-\triangle)$. Suppose $u \in H^1(\Omega) \cap C(\Omega)$ satisfies $-\triangle u + \lambda u \geq 0$ (respectively, $\leq 0$) in $H^{-1}(\Omega)$. If $u^- \in H_0^1(\Omega)$ (respectively, $u^+ \in H_0^1(\Omega)$) and if $u \not\equiv 0$, then $u > 0$ (respectively, $u < 0$) in $\Omega$.*

The proof of Theorem 3.1.2 is based on the following simple lemma.

LEMMA 3.1.3. *Let $0 < \rho < R < \infty$ and set $\omega = \{\rho < |x| < R\}$. Let $\lambda \in \mathbb{R}$ and suppose $\beta > \max\{0, N-2\}$ satisfies $\beta(\beta - N + 2) \geq |\lambda| R^2$. If $v$ is defined by $v(x) = |x|^{-\beta} - R^{-\beta}$ for $\rho \leq |x| \leq R$, then the following properties hold.*

(i) $v \in C^\infty(\overline{\omega})$.
(ii) $v(x) = 0$ *if* $|x| = R$.
(iii) $\rho^{-\beta} > v(x) \geq \beta R^{-(\beta+1)}(R - |x|)$ *if* $\rho \leq |x| \leq R$.
(iv) $-\Delta v + \lambda v \leq 0$ *in* $\omega$.

Proof. Properties (i), (ii) and (iii) are immediate. Next,

$$\begin{aligned}-\Delta v+\lambda v&=-\beta(\beta-N+2)|x|^{-(\beta+2)}+\lambda|x|^{-\beta}-\lambda R^{-\beta}\\&\le-\beta(\beta-N+2)R^{-2}|x|^{-\beta}+|\lambda|\,|x|^{-\beta},\end{aligned}$$

and (iv) easily follows. □

Proof of Theorem 3.1.2. We prove the first part of the result, the second follows by changing $u$ to $-u$. We first note that by Theorem 3.1.1, $u\ge 0$ a.e. in $\Omega$. Since $u\in C(\Omega)$ and $u\not\equiv 0$, the set

$$O=\{x\in\Omega;\ u(x)>0\},$$

is a nonempty open subset of $\Omega$. $\Omega$ being connected, we need only show that $O$ is a closed subset of $\Omega$. Suppose $(y_n)_{n\ge 0}\subset O$ and $y_n\to y\in\Omega$ as $n\to\infty$. Let $R>0$ be such that $B(y,2R)\subset\Omega$, and fix $n_0$ large enough so that $|y-y_{n_0}|<R$. Since $u(y_{n_0})>0$, there exist $0<\rho<R$ and $\varepsilon>0$ such that $u(x)\ge\varepsilon$ for $|x-y_{n_0}|=\rho$. Set $U=\{\rho<|x-y_{n_0}|<R\}$ and let $w(x)=u(x)-\varepsilon\rho^\beta v(x-y_{n_0})$ for $x\in U$, where $\beta$ and $v$ are as in Lemma 3.1.3. It follows that $w\in H^1(U)\cap C(\overline{U})$. Moreover, $-\Delta w+\lambda w\ge 0$ by property (iv) of Lemma 3.1.3. Also, since $u\ge 0$ in $\Omega$, we deduce from property (ii) of Lemma 3.1.3 that $w(x)\ge 0$ if $|x-y_{n_0}|=R$. Furthermore, $w(x)\ge 0$ if $|x-y_{n_0}|=\rho$ by property (iii) of Lemma 3.1.3 and because $u(x)\ge\varepsilon$. Thus we may apply Theorem 3.1.1 and we deduce that $w(x)\ge 0$ for $x\in U$. In particular, $u(y)\ge\varepsilon\rho^\beta v(y-y_{n_0})>0$ by property (iii) of Lemma 3.1.3, so that $y\in O$. Therefore, $O$ is closed, which completes the proof. □

We now state a stronger version of the maximum principle, which requires a certain amount of regularity of the domain.

Theorem 3.1.4. *Let $\Omega\subset\mathbb{R}^N$ be a bounded, connected, open set. Assume there exist $\eta,\nu>0$ with the following property. For every $x\in\Omega$ such that $\mathrm{d}(x,\partial\Omega)\le\eta$, there exists $y\in\Omega$ such that*

$$\begin{cases}x\in B(y,\eta),\\ B(y,\eta)\subset\Omega,\\ \eta-|x-y|\ge\nu\mathrm{d}(x,\partial\Omega).\end{cases}\tag{3.1.1}$$

*Let $\lambda_1(-\triangle)$ be defined by (2.1.5) and let $\lambda>-\lambda_1(-\triangle)$. Suppose $u\in H^1(\Omega)\cap C(\Omega)$ satisfies $-\triangle u+\lambda u\ge 0$ (respectively, $\le 0$) in $H^{-1}(\Omega)$. If $u^-\in H^1_0(\Omega)$ (respectively, $u^+\in H^1_0(\Omega)$) and if $u\not\equiv 0$, then there exists $\mu>0$ such that $u(x)\ge\mu\mathrm{d}(x,\partial\Omega)$ (respectively, $u(x)\le-\mu\mathrm{d}(x,\partial\Omega)$) in $\Omega$.*

Remark 3.1.5. The assumption (3.1.1) is satisfied if $\partial\Omega$ is of class $C^2$. Indeed, let $\gamma(z)$ denote the unit outwards normal to $\partial\Omega$ at $z\in\partial\Omega$. Since $\Omega$ is bounded, $\partial\Omega$ is uniformly $C^2$, so that there exists $\eta>0$ such that $B(z-\eta\gamma(z),\eta)\subset\Omega$ for every $z\in\partial\Omega$. If $x\in\Omega$ and $\mathrm{d}(x,\partial\Omega)\le\eta$, let $z\in\partial\Omega$ be such that $|x-z|=\mathrm{d}(x,\partial\Omega)$. It follows that $x-z$ is parallel to $\gamma(y)$. Thus, if we set $y=z-\eta\gamma(z)$, we see that $x\in B(y,\eta)$, $B(y,\eta)\subset\Omega$ and $\eta-|x-y|=|z-x|=\mathrm{d}(x,\partial\Omega)$.

Proof of Theorem 3.1.4. We prove the first part of the result, the second follows by changing $u$ to $-u$. Let $0<\varepsilon\le\eta/2$ and consider $\Omega_\varepsilon=\{x\in\Omega;\ \mathrm{d}(x,\partial\Omega)\ge\varepsilon\}$. We fix $\varepsilon>0$ sufficiently small so that $\Omega_\varepsilon$ is a nonempty, compact subset of $\Omega$. It follows from Theorem 3.1.2 that there exists $\delta>0$ such that

$$u(x)\ge\delta\quad\text{for all}\quad x\in\Omega_\varepsilon.\tag{3.1.2}$$

We now consider $x_0\in\Omega$ such that $\mathrm{d}(x_0,\partial\Omega)<\varepsilon$, and we let $y_0\in\Omega$ satisfy (3.1.1). Since $B(y_0,\eta)\subset\Omega$ and $\eta\ge 2\varepsilon$, we see that $\mathrm{d}(z,\partial\Omega)\ge\varepsilon$ for all $z\in B(y_0,\eta/2)$. It then follows from (3.1.2) that

$$u(z) \geq \delta \quad \text{for all} \quad z \in B(y_0, \eta/2). \tag{3.1.3}$$

We let $\rho = \eta/2$, $R = \eta$ and $U = \{\rho < |x - y_0| < R\}$, so that $x_0 \in U$. Let $w(x) = u(x) - \varepsilon\rho^\beta v(x - y_0)$ for $x \in U$, where $\beta$ and $v$ are as in Lemma 3.1.3. It follows that $w \in H^1(U) \cap C(\overline{U})$. Moreover, $-\Delta w + \lambda w \geq 0$ by property (iv) of Lemma 3.1.3. Also, $w(x) \geq 0$ if $|x - y_0| = R$ by property (ii) of Lemma 3.1.3 and because $u \geq 0$ in $\Omega$. Furthermore, $w(x) \geq 0$ if $|x - y_0| = \rho$ by property (iii) of Lemma 3.1.3 and (3.1.3). Thus we may apply Theorem 3.1.1 and we deduce that $w(x) \geq 0$ for $x \in U$. In particular,

$$u(x_0) \geq \beta\varepsilon\rho^\beta R^{-(\beta+1)}(R - |x_0 - y_0|) \geq \nu\beta\varepsilon\rho^\beta R^{-(\beta+1)}\mathrm{d}(x_0, \partial\Omega),$$

where the first inequality above follows from of Lemma 3.1.3 (iii) and the second from (3.1.1). Since $x_0 \in \Omega \setminus \Omega_\varepsilon$ is arbitrary, we see that there exists $\mu > 0$ such that $u(x) \geq \mu \mathrm{d}(x, \partial\Omega)$ for all $x \in \Omega \setminus \Omega_\varepsilon$. On the other hand, (3.1.2) implies that there exists $\mu' > 0$ such that $u(x) \geq \mu' \mathrm{d}(x, \partial\Omega)$ for all $x \in \Omega_\varepsilon$. This completes the proof. □

## 3.2. The spectral decomposition of the Laplacian

Throughout this section, we assume that $\Omega$ is bounded and connected, and we give some important properties concerning the spectral decomposition of $-\triangle + a$ where $a \in L^\infty(\Omega)$.

THEOREM 3.2.1. *Assume $\Omega$ is a bounded, connected domain of $\mathbb{R}^N$ and let $a \in L^\infty(\Omega)$. It follows that there exist a nondecreasing sequence $(\lambda_n)_{n\geq 1} \subset \mathbb{R}$ with $\lambda_n \to \infty$ as $n \to \infty$ and a Hilbert basis $(\varphi_n)_{n\geq 1}$ of $L^2(\Omega)$ such that $(\varphi_n)_{n\geq 1} \subset H^1_0(\Omega)$ and*

$$-\Delta\varphi_n + a\varphi_n = \lambda_n\varphi_n, \tag{3.2.1}$$

*in $H^{-1}(\Omega)$. Moreover, the following properties hold.*

(i) *$\varphi_n \in L^\infty(\Omega) \cap C(\Omega)$, for every $n \geq 1$.*
(ii) *$\lambda_1 = \lambda_1(-\Delta + a; \Omega)$, where $\lambda_1(-\Delta + a; \Omega)$ is defined by (2.1.5).*
(iii) *$\lambda_1$ is a simple eigenvalue and either $\varphi_1 > 0$ or else $\varphi_1 < 0$ on $\Omega$.*

For the proof of Theorem 3.2.1, we will use the following fundamental property.

PROPOSITION 3.2.2. *Suppose $\Omega$ is a bounded, connected domain of $\mathbb{R}^N$. Let $a \in L^\infty(\Omega)$ and $\Lambda = \lambda_1(-\Delta + a)$ where $\lambda_1(-\Delta + a)$ is defined by (2.1.5). It follows that there exists $\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$, $\varphi > 0$ in $\Omega$, $\|\varphi\|_{L^2} = 1$, such that*

$$-\Delta\varphi + a\varphi = \Lambda\varphi, \tag{3.2.2}$$

*in $H^{-1}(\Omega)$. In addition, the following properties hold.*

(i) *$\varphi$ is the unique nonnegative solution of the minimization problem*

$$u \in S, \quad J(u) = \inf_{v\in S} J(v), \tag{3.2.3}$$

*where*

$$J(v) = \frac{1}{2}\int_\Omega \{|\nabla v|^2 + av^2\}, \tag{3.2.4}$$

*and $S = \{v \in H^1_0(\Omega); \|v\|_{L^2} = 1\}$.*

(ii) *If $\psi \in H^1_0(\Omega)$ is a solution of the equation (3.2.2), then there exists a constant $c \in \mathbb{R}$ such that $\psi = c\varphi$.*

PROOF. We proceed in five steps.

STEP 1. The minimization problem (3.2.3) has a solution $u \in H^1_0(\Omega)$, $u \geq 0$. Indeed, it follows from the techniques of Section 2.2 that if $J$ is defined by (3.2.4), then $J \in C^1(H^1_0(\Omega), \mathbb{R})$ and $J'(u) = -\Delta u + au$ for all $u \in H^1_0(\Omega)$. Moreover, if

$F(u) = \|u\|_{L^2}^2$, then $F \in C^1(H_0^1(\Omega), \mathbb{R})$ and $F'(u) = 2u$ for all $u \in H_0^1(\Omega)$. Next, it follows from (2.1.5) that

$$\inf_{v \in S} J(v) = \frac{\Lambda}{2}. \tag{3.2.5}$$

Consider a minimizing sequence $(v_n)_{n\geq 0}$ of (3.2.3). Letting $u_n = |v_n|$, it follows that $u_n \geq 0$ and that $(u_n)_{n\geq 0}$ is also a minimizing sequence of (3.2.3). Moreover, since $\|u_n\|_{L^2} = 1$, we see that $(u_n)_{n\geq 0}$ is bounded in $H_0^1(\Omega)$. Since $\Omega$ is bounded, it follows that there exists $u \in H_0^1(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$ and $\|\nabla u\|_{L^2} \leq \liminf \|\nabla u_n\|_{L^2}$ as $n \to \infty$ (see Theorem 5.5.5). In particular, $u \geq 0$ and $u$ is a solution of the minimization problem (3.2.3).

STEP 2. If $u \in H_0^1(\Omega)$ and $\|u\|_{L^2} = 1$, then is a solution of the minimization problem (3.2.3) if and only if $u$ is a solution of the equation (3.2.2). Suppose first that $u$ is a solution of the minimization problem (3.2.3). It follows from Theorem 2.4.1 that there exists a Lagrange multiplier $\lambda$ such that $-\Delta u + au = \lambda u$. Taking the $H^{-1} - H_0^1$ duality product of that equation with $u$, we see that $2J(u) = \lambda$, so that $\lambda = \Lambda$ by (3.2.5). Thus $u$ satisfies the equation (3.2.2). Conversely, suppose $u$ is a solution of the equation (3.2.2). Taking the $H^{-1} - H_0^1$ duality product of the equation with $u$, we see that $2J(u) = \Lambda$, so that, by (3.2.5), $u$ is a solution of the minimization problem (3.2.3).

STEP 3. If $u \in H_0^1(\Omega)$, $u \geq 0$, $u \not\equiv 0$, is a solution of the equation (3.2.2), then $u \in L^\infty(\Omega) \cap C(\Omega)$ and $u > 0$ in $\Omega$. The property $u \in L^\infty(\Omega) \cap C(\Omega)$ follows from Corollary 4.4.3. Next, we write (3.2.2) in the form $-\Delta u + \|a\|_{L^\infty} u = (\|a\|_{L^\infty} - a)u$. Since $\|a\|_{L^\infty} - a \geq 0$, we see that $(\|a\|_{L^\infty} - a)u \geq 0$, and it follows from the strong maximum principle that $u > 0$ in $\Omega$.

STEP 4. If $u, v \geq 0$ are two solutions of the minimization problem (3.2.3), then $u = v$. Indeed, let $w = u - v$ and assume by contradiction that $w \not\equiv 0$. It follows from Step 2 that $w$ is a solution of the equation (3.2.2). Thus, again by Step 2, $w/\|w\|_{L^2}$ is a solution of the minimization problem (3.2.3), so that $z = |w|/\|w\|_{L^2}$ is also a solution of the minimization problem (3.2.3). By Steps 2 and 3, we see that $z > 0$ in $\Omega$, so that $w$ does not vanish in $\Omega$. In particular, $w$ does not change sign and we may assume for example that $w > 0$. It follows that $0 \leq v < u$, which is absurd since $\|u\|_{L^2} = \|v\|_{L^2}$.

STEP 5. Conclusion. Let $\varphi = u$ with $u$ as in Step 1. It follows from Steps 2 and 3 that $\varphi$ is a solution of (3.2.2) and that $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$, $\varphi > 0$ in $\Omega$ and $\|\varphi\|_{L^2} = 1$. Next, property (i) follows from Step 4. Finally, suppose $\psi \in H_0^1(\Omega)$, $\psi \not\equiv 0$, is a solution of the equation (3.2.2). Setting $z = |\psi|/\|\psi\|_{L^2}$, we deduce from Step 2 (see the proof of Step 4) that $z$ is a nonnegative solution of (3.2.3). Thus $z = \varphi$, so that $|\psi| = \|\psi\|_{L^2}\varphi$. In particular, $\psi$ does not vanish in $\Omega$, so that $\psi$ has constant sign. We deduce that $\psi = \pm\|\psi\|_{L^2}\varphi$, which proves property (ii). □

PROOF OF THEOREM 3.2.1. Set $\Lambda = \lambda_1(-\Delta + a)$ where $\lambda_1(-\Delta + a)$ is defined by (2.1.5). Let $f \in H^{-1}(\Omega)$, and let $u \in H_0^1(\Omega)$ be the solution of the equation $-\triangle u + au + (1 - \Lambda)u = f$ in $H^{-1}(\Omega)$. Let us set $u = Kf$. By Theorem 2.1.4, $K$ is bounded $H^{-1}(\Omega) \to H_0^1(\Omega)$, hence $L^2(\Omega) \to H_0^1(\Omega)$. Therefore, by Theorem 5.5.4, $K$ is compact $L^2(\Omega) \to L^2(\Omega)$. We claim that $K$ is self adjoint. Indeed, let $f, g \in L^2(\Omega)$ and let $u = Kf$, $v = Kg$. We have

$$\begin{aligned}(u, g)_{L^2} - (f, v)_{L^2} = {} & (-\triangle v + av + (1 - \Lambda)v, u)_{H^{-1}, H_0^1} \\ & - (-\triangle u + au + (1 - \Lambda)u, v)_{H^{-1}, H_0^1} = 0,\end{aligned}$$

by (5.1.5). It is clear that $K^{-1}(0) = \{0\}$. Moreover, if $f \in L^2(\Omega)$ and $u = Kf$, then

$$(Kf, f)_{L^2} = (u, -\Delta u + au + (1 - \Lambda)u)_{H_0^1, H^{-1}} = \int_\Omega (|\nabla u|^2 + au^2 + (1 - \Lambda)u^2),$$

so that by (2.1.6),

$$(Kf, f)_{L^2} \ge \int_\Omega u^2 \ge 0.$$

Therefore (see Brezis [11], Theorem VI.11), $L^2(\Omega)$ possesses a Hilbert basis $(\varphi_n)_{n\ge1}$ made of eigenvectors of $K$, and the eigenvalues of $K$ consist of a nonincreasing sequence $(\sigma_n)_{n\ge1} \subset (0,\infty)$ converging to 0, as $n \to \infty$. We observe that $\varphi_n = \sigma_n^{-1} K\varphi_n$, so that $\varphi_n \in H_0^1(\Omega)$ and $\varphi_n$ satisfies the equation (3.2.1) with

$$\lambda_n = \frac{1}{\sigma_n} - 1 + \Lambda. \tag{3.2.6}$$

This proves the first statement, and we now prove properties (i), (ii) and (iii).

(i) This follows from Corollary 4.4.3.

(ii) By formula (3.2.6), this amounts in showing that $\sigma_1 = 1$. We first observe that if $\varphi$ is as in Proposition 3.2.2, then $K\varphi = \varphi$. Thus 1 is an eigenvalue of $K$, so that $\sigma_1 \ge 1$. Next, we see that $-\Delta\varphi_1 + a\varphi_1 = (\sigma_1^{-1} - 1 + \Lambda)\varphi_1$. Taking the $H^{-1} - H_0^1$ duality product of the equation with $\varphi_1$, we deduce that

$$\int_\Omega \{|\nabla\varphi_1|^2 + a\varphi_1^2\} = \sigma_1^{-1} - 1 + \Lambda.$$

Using (2.1.5), we deduce that $\sigma_1^{-1} - 1 + \Lambda \ge \Lambda$. Thus $\sigma_1 \le 1$, which proves (ii).

(iii) This follows from Proposition 3.2.2. □

REMARK 3.2.3. Connectedness of $\Omega$ is required only for property (iii) of Theorem 3.2.1. Without Connectedness, these two properties may not hold, as shows the following example. Let $\Omega = (0,\pi) \cup (\pi, 2\pi)$. Then $\lambda_1 = 1$, and the corresponding eigenspace is two-dimensional. More precisely, it is the spaces spanned by the two functions $\varphi_1$ and $\widetilde{\varphi}_1$ defined by

$$\varphi_1(x) = \begin{cases} \sin x \text{ if } 0 < x < \pi, \\ 0 \text{ if } \pi < x < 2\pi, \end{cases} \qquad \widetilde{\varphi}_1(x) = \begin{cases} 0 \text{ if } 0 < x < \pi, \\ -\sin x \text{ if } \pi < x < 2\pi. \end{cases}$$

In particular, both $\varphi_1$ and $\widetilde{\varphi}_1$ vanish on a connected component of $\Omega$.

We end this section with a useful characterization of $H_0^1(\Omega)$ in terms of the spectral decomposition.

PROPOSITION 3.2.4. *Assume $\Omega$ is a bounded, connected domain of $\mathbb{R}^N$, let $a \in L^\infty(\Omega)$ and let $(\lambda_n)_{n\ge1} \subset \mathbb{R}$ and $(\varphi_n)_{n\ge1} \subset H_0^1(\Omega)$ be as in Theorem 3.2.1. Given any $u \in L^2(\Omega)$, let $\alpha_j = (u, \varphi_j)_{L^2}$ for all $j \ge 1$ so that $u = \sum \alpha_j\varphi_j$. It follows that $u \in H_0^1(\Omega)$ if and only if $\sum \lambda_j\alpha_j^2 < \infty$. Moreover, $\sum \lambda_j\alpha_j^2 = \|\nabla u\|_{L^2}^2$.*

PROOF. Since $(\varphi_j)_{j\ge1}$ is a Hilbert basis of $L^2(\Omega)$, we may consider the isometric isomorphism $T : \ell^2(\mathbb{N}) \to L^2(\Omega)$ defined by

$$TA = \sum_{j=1}^\infty \alpha_j\varphi_j,$$

if $A = (\alpha_j)_{j\ge1}$. Let now

$$\mathcal{V} = \{A \in \ell^2(\mathbb{N}); \sum \lambda_j\alpha_j^2 < \infty\},$$

equipped with the norm

$$\|A\|_{\mathcal{V}} = \Big(\sum_{j=1}^{\infty} \lambda_j \alpha_j^2\Big)^{\frac{1}{2}},$$

so that $\mathcal{V}$ is a Banach (in fact, Hilbert) space. We first claim that $T(\mathcal{V}) \subset H_0^1(\Omega)$ and

$$\|A\|_{\mathcal{V}} = \|\nabla TA\|_{L^2}, \tag{3.2.7}$$

for all $A \in \mathcal{V}$. Indeed, let $A \in \mathcal{V}$ and consider the sequence $(A_n)_{n\ge 0} \subset \mathcal{V}$ defined by $\alpha_{n,j} = \alpha_j$ if $j \le n$ and $\alpha_{n,j} = 0$ if $j > n$. It follows that $A_n \to A$ in $\mathcal{V}$ (hence in $\ell^2(\mathbb{N})$). Moreover,

$$\begin{aligned}
\|\nabla TA_n\|_{L^2}^2 &= (-\Delta TA_n, TA_n)_{H^{-1},H_0^1} \\
&= \Big(\sum_{j=1}^{n} \lambda_j \alpha_j \varphi_j, \sum_{j=1}^{n} \alpha_j \varphi_j\Big)_{H^{-1},H_0^1} \\
&= \Big(\sum_{j=1}^{n} \lambda_j \alpha_j \varphi_j, \sum_{j=1}^{n} \alpha_j \varphi_j\Big)_{L^2} \\
&= \sum_{j=1}^{n} \lambda_j \alpha_j^2 = \|A_n\|_{\mathcal{V}}^2.
\end{aligned} \tag{3.2.8}$$

A similar calculation shows that if $m > n \ge 1$, then

$$\|\nabla(TA_m - TA_n)\|_{L^2}^2 = \sum_{j=n+1}^{m} \lambda_j \alpha_j^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{3.2.9}$$

We deduce in particular from (3.2.9) that $(TA_n)_{n\ge1}$ is a Cauchy sequence in $H_0^1(\Omega)$. Since $TA_n \to TA$ in $L^2(\Omega)$ (because $A_n \to A$ in $\ell^2(\mathbb{N})$), we deduce that $TA \in H_0^1(\Omega)$ and $TA_n \to TA$ in $\mathcal{V}$. Thus (3.2.7) follows by letting $n \to \infty$ in (3.2.8). It now remains to show that $T\mathcal{V} = H_0^1(\Omega)$. Since $\mathcal{V}$ is a Banach space and $T : \mathcal{V} \to H_0^1(\Omega)$ is isometric, we see that $T\mathcal{V}$ is a closed subspace of $H_0^1(\Omega)$. Suppose by contradiction that $\mathcal{V} \ne H_0^1(\Omega)$. It follows that there exists $w \in H_0^1(\Omega)$, $w \ne 0$ such that $(TA, w)_{H_0^1} = 0$ for all $A \in \mathcal{V}$. Fix $n \ge 1$, and define $A \in \mathcal{V}$ by $\alpha_j = 1$ if $j = n$ and $\alpha_j = 0$ if $j \ne n$. It follows that $TA = \varphi_n$, so that $(\varphi_n, w)_{H_0^1} = 0$. Since

$$\begin{aligned}
(\varphi_n, w)_{H_0^1} &= (\nabla\varphi_n, \nabla w)_{L^2} = (-\Delta\varphi_n, w)_{H^{-1},H_0^1} \\
&= \lambda_n(\varphi_n, w)_{H^{-1},H_0^1} = \lambda_n(\varphi_n, w)_{L^2},
\end{aligned}$$

and $n \ge 1$ is arbitrary, we conclude that $w = 0$, which is a contradiction. □

## 3.3. The iteration method

Throughout this section, we assume that $\Omega$ is a bounded domain of $\mathbb{R}^N$ and we consider the problem

$$\begin{cases} -\triangle u = g(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.3.1}$$

where $g \in C^1(\mathbb{R})$ is a given nonlinearity. The method we will use relies on the notion of sub- and supersolutions, which are defined as follows.

Definition 3.3.1. A function $\overline{u} \in H^1(\Omega) \cap L^\infty(\Omega)$ is called a supersolution of (3.3.1) if the following properties hold.

(i) $-\triangle\overline{u} \ge g(\overline{u})$ in $H^{-1}(\Omega)$;
(ii) $\overline{u}^- \in H_0^1(\Omega)$.

Similarly, a function $\underline{u} \in H^1(\Omega) \cap L^\infty(\Omega)$ is called a subsolution of (3.3.1) if the following properties hold.

(iii) $-\triangle \underline{u} \le g(\underline{u})$ in $H^{-1}(\Omega)$;
(iv) $\underline{u}^+ \in H_0^1(\Omega)$.

In particular, a solution $u \in H_0^1(\Omega)$ of (3.3.1) is a both a supersolution and a subsolution.

REMARK 3.3.2. Here are some comments on Definition 3.3.1.

(i) As observed in Section 3.1, the property $-\triangle \overline{u} \ge g(\overline{u})$ in $H^{-1}(\Omega)$ is equivalent to saying that $-\triangle \overline{u} \ge g(\overline{u})$ in $\mathcal{D}'(\Omega)$, i.e. that $(-\triangle \overline{u} - g(\overline{u}), \varphi)_{\mathcal{D}',\mathcal{D}} \ge 0$ for all $\varphi \in C_c^\infty(\Omega)$, $\varphi \ge 0$.
(ii) It follows from (i) above that if $\overline{u} \in L^\infty(\Omega) \cap H^2(\Omega)$ satisfies $-\triangle \overline{u} \ge g(\overline{u})$ a.e. in $\Omega$, then $-\triangle \overline{u} \ge g(\overline{u})$ in $H^{-1}(\Omega)$.
(iii) Property (ii) of Definition 3.3.1 is a weak way of saying "$\overline{u} \ge 0$ on $\partial\Omega$". Indeed, if $\overline{u} \in C(\overline{\Omega})$, then $\overline{u} \ge 0$ on $\partial\Omega$ implies that $\overline{u}^- \in H_0^1(\Omega)$ (see Remark 5.1.10 (ii)). Conversely, if $\Omega$ is of class $C^1$, if $\overline{u} \in C(\overline{\Omega})$ and if $\overline{u}^- \in H_0^1(\Omega)$, then $\overline{u} \ge 0$ on $\partial\Omega$ (see Remark 5.1.10 (iii)). (A similar observation holds for property (iv).)

Our main result of this section is the following.

THEOREM 3.3.3. *Assume that $\Omega$ is a bounded domain of $\mathbb{R}^N$ and let $g \in C^1(\mathbb{R})$. Suppose that there exist a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (3.3.1). If $\underline{u} \le \overline{u}$ a.e. in $\Omega$, then the following properties hold.*

(i) *There exists a solution $\widetilde{u} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (3.3.1) which is* minimal *with respect to $\underline{u}$, in the sense that if $w$ is any supersolution of (3.3.1) with $w \ge \underline{u}$, then $w \ge \widetilde{u}$.*
(ii) *Similarly, there exists a solution $\widehat{u} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (3.3.1) which is* maximal *with respect to $\overline{u}$, in the sense that if $z$ is any subsolution of (3.3.1) with $z \le \overline{u}$, then $z \le \widehat{u}$.*
(iii) *In particular, $\underline{u} \le \widetilde{u} \le \widehat{u} \le \overline{u}$. (Note that $\widetilde{u}$ and $\widehat{u}$ may coincide.)*

REMARK 3.3.4. Here are some comments on Theorem 3.3.3.

(i) The main conclusion of Theorem 3.3.3 is the existence of a solution of (3.3.1). On the other hand, the maximal and minimal solutions can be useful.
(ii) Theorem 3.3.3 is somewhat surprising because no assumption is made on the behavior of $g$. Of course, in practice the behavior of $g$ will be important for the construction of sub- and supersolutions.
(iii) The assumption $\underline{u} \le \overline{u}$ is absolutely essential in Theorem 3.3.3. This can be seen on a quite elementary example: consider for example the equation

$$\begin{cases} -u'' = 2 + 9u, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{3.3.2}$$

in $\Omega = (0, \pi)$. It is clear that $\underline{u}(x) \equiv 0$ is a subsolution. Furthermore, $\overline{u}(x) = -\sin(x)^2$ is a supersolution. Indeed,

$$-\overline{u}'' - 9\overline{u} = 2 + 5\sin(x)^2 \ge 2.$$

Next, we claim that there is no solution of (3.3.2). Indeed, suppose $u$ satisfies (3.3.2). Multiplying the equation by $\sin(3x)$ and integrating by parts, we obtain

$$9\int_0^\pi u(x)\sin(3x) = 2\int_0^\pi \sin(3x) + 9\int_0^\pi u(x)\sin(3x),$$

which is absurd since

$$\int_0^\pi \sin(3x) = \frac{2}{3} \ne 0.$$

Thus we have an example where there is a subsolution, there is a supersolution, but there is no solution. Obviously, Theorem 3.3.3 does not apply because $\underline{u} \not\leq \overline{u}$.

Proof of Theorem 3.3.3. We use an iteration technique. Set

$$M = \max\{\|\underline{u}\|_{L^\infty}, \|\overline{u}\|_{L^\infty}\},$$

let

$$\lambda \geq \|g'\|_{L^\infty(-M,M)}, \tag{3.3.3}$$

and set

$$g_\lambda(u) = g(u) + \lambda u, \tag{3.3.4}$$

for all $u \in \mathbb{R}$. Finally, we set $u_0 = \underline{u}$ and $u^0 = \overline{u}$ and we define the sequences $(u_n)_{n\geq 0}$ and $(u^n)_{n\geq 0}$ by induction as follows.

$$\begin{cases} -\triangle u_{n+1} + \lambda u_{n+1} = g_\lambda(u_n), \\ u_{n+1|\partial\Omega} = 0, \end{cases} \qquad \begin{cases} -\triangle u^{n+1} + \lambda u^{n+1} = g_\lambda(u^n), \\ u^{n+1}{}_{|\partial\Omega} = 0, \end{cases} \tag{3.3.5}$$

We will show that the sequences are well defined, that

$$u_0 \leq u_1 \leq u_2 \leq \cdots \leq u^2 \leq u^1 \leq u^0,$$

and that $\widetilde{u} = \lim\limits_{n\to\infty} u_n$ and $\widehat{u} = \lim\limits_{n\to\infty} u^n$ have the desired properties. We proceed in five steps.

Step 1. $u_0 \leq u_1 \leq u^1 \leq u^0$. Since $u_0, u^0 \in L^\infty(\Omega)$, we see that $g_\lambda(u_0), g_\lambda(u^0) \in L^\infty(\Omega)$. Therefore, we may apply Theorem 2.1.4, from which it follows that $u_1, u^1 \in H^1_0(\Omega)$ are well-defined. (Note that $\lambda \geq 0$ and, since $\Omega$ is bounded, $\lambda_1(-\Delta) > 0$, see Remark 2.1.5.) Furthermore,

$$-\triangle u_1 + \lambda u_1 = g_\lambda(u_0) \geq -\triangle u_0 + \lambda u_0,$$

since $u_0$ is a subsolution. It follows from the maximum principle that $u_1 \geq u_0$. One shows similarly that $u^1 \leq u^0$. Next, observe that $g_\lambda$ is nondecreasing on $[-M, M]$; and so,

$$-\triangle u_1 + \lambda u_1 = g_\lambda(u_0) \leq g_\lambda(u^0) = -\triangle u^1 + \lambda u^1.$$

Applying again the maximum principle, we deduce that $u_1 \leq u^1$. Hence the result.

Step 2. For all $n \geq 1$, $u_n$ and $u^n$ are well-defined, and $u_{n-1} \leq u_n \leq u^n \leq u^{n-1}$. We argue by induction. It follows from Step 1 that the property holds for $n = 1$. Suppose it holds up to some $n \geq 1$. In particular, $u_n, u^n \in L^\infty(\Omega)$, so that $u_{n+1}, u^{n+1} \in H^1_0(\Omega)$ are well-defined by Theorem 2.1.4. Next, since $g_\lambda$ is nondecreasing on $[-M, M]$, it follows that

$$-\triangle u_{n+1} + \lambda u_{n+1} = g_\lambda(u_n) \geq g_\lambda(u_{n-1}) = -\triangle u_n + \lambda u_n;$$

and so, $u_{n+1} \geq u_n$ by the maximum principle. One shows as well that $u^{n+1} \leq u^n$. Finally, using again the nondecreaing character of $g_\lambda$, we see that

$$-\triangle u_{n+1} + \lambda u_{n+1} = g_\lambda(u_n) \leq g_\lambda(u^n) = -\triangle u^{n+1} + \lambda u^{n+1};$$

and so, $u_{n+1} \leq u^{n+1}$ by the maximum principle. Thus $u_n \leq u_{n+1} \leq u^{n+1} \leq u^n$, which proves the result.

Step 3. If $\widetilde{u} = \lim u_n$ and $\widehat{u} = \lim u^n$ as $n \to \infty$, then $\widetilde{u}, \widehat{u} \in H^1_0(\Omega) \cap L^\infty(\Omega)$ are solutions of (3.3.1). Indeed, it follows from Step 2 that $(u_n)_{n\geq 0}$ is nondecreasing and bounded in $L^\infty(\Omega)$. Thus $\widetilde{u} = \lim u_n$ as $n \to \infty$ is well-defined as a limit a.e. in $\Omega$. Since $g_\lambda$ is continuous, it follows that $g_\lambda(u_n) \to g_\lambda(\widetilde{u})$ a.e. in $\Omega$. Since $(u_n)_{n\geq 0}$ is bounded in $L^\infty(\Omega)$, $(g_\lambda(u_n))_{n\geq 0}$ is also bounded in $L^\infty(\Omega)$, and by the dominated convergence theorem it follows that $g(u_n) \to g_\lambda(\widetilde{u})$ in $L^2(\Omega)$.

Finally, since $(-\triangle + \lambda I)^{-1}$ is continuous $L^2(\Omega) \to H_0^1(\Omega)$ (see Theorem 2.1.4), we deduce that $u_n$ converges in $H_0^1(\Omega)$ as $n \to \infty$ to the solution $v \in H_0^1(\Omega)$ of

$$\begin{cases} -\triangle v + \lambda v = g_\lambda(\widetilde{u}), \\ v_{|\partial\Omega} = 0. \end{cases}$$

Since $u_n \to \widetilde{u}$ a.e., we see that $v = \widetilde{u}$ and the result follows for $\widetilde{u}$. A similar argument applies to $\widehat{u}$.

STEP 4. The solutions $\widetilde{u}$ and $\widehat{u}$ are independent of $\lambda$ satisfying (3.3.3). We show the result for $\widetilde{u}$, and the same argument applies to $\widehat{u}$. Let $\lambda, \lambda'$ satisfy (3.3.3) and let $\widetilde{u} = \lim\limits_{n\to\infty} u_n$ and $\widetilde{u}' = \lim\limits_{n\to\infty} u'_n$ be the corresponding solutions of (3.3.1) constructed as above. Since $\lambda$ and $\lambda'$ play a similar role, we need only show that We first show that $\widetilde{u} \le \widetilde{u}'$. Assume for definiteness that $\lambda \ge \lambda'$. We claim that $\widetilde{u}' \ge u_n$ for all $n \ge 0$. We argue by induction. This is clear for $n = 0$. Assuming it holds for some $n \ge 0$, we have

$$-\triangle\widetilde{u}' + \lambda\widetilde{u}' = g_\lambda(\widetilde{u}') \ge g_\lambda(u_n) = -\triangle u_{n+1} + \lambda u_{n+1}.$$

It follows from the maximum principle that $\widetilde{u}' \ge u_{n+1}$, which proves the claim. Letting $n \to \infty$, we deduce that $\widetilde{u} \le \widetilde{u}'$.

STEP 5. Minimality of $\widetilde{u}$ and maximality of $\widehat{u}$. We only show the minimality of $\widetilde{u}$, the other argument being similar. Let $w$ be a supersolution of (3.3.1), $w \ge u_0$. Let $\widetilde{M} = \max\{\|\underline{u}\|_{L^\infty}, \|\overline{u}\|_{L^\infty}, \|w\|_{L^\infty}\}$ and let

$$\lambda \ge \|g'\|_{L^\infty(-\widetilde{M},\widetilde{M})}.$$

Let $(u_n)_{n\ge0}$ be the corresponding sequence defined by (3.3.5), so that $\widetilde{u} = \lim\limits_{n\to\infty} u_n$ by Steps 3 and 4. We claim that $w \ge u_n$ for all $n \ge 0$. We argue by induction. This is true by assumption for $n = 0$. Assuming this holds up to some $n \ge 0$, we have

$$-\triangle w + \lambda w \ge g_\lambda(w) \ge g_\lambda(u_n) = -\triangle u_{n+1} + \lambda u_{n+1}.$$

It follows from the maximum principle that $w \ge u_{n+1}$, which proves the claim. We deduce the result by letting $n \to \infty$. This completes the proof. □

We now give applications of Theorem 3.3.3 in elementary cases where sub- and supersolutions are trivially constructed. We will consider more delicate cases in the next section.

COROLLARY 3.3.5. *Assume that $\Omega$ is a bounded domain of $\mathbb{R}^N$ and let $g \in C^1(\mathbb{R})$. If there exist $a \le 0 \le b$ such that $g(a) = g(b) = 0$, then there exists a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (3.3.1), with $a \le u \le b$.*

PROOF. $\underline{u} \equiv a$ is clearly a subsolution and $\overline{u} \equiv b$ is clearly a supersolution. □

COROLLARY 3.3.6. *If $\Omega$ is a bounded domain of $\mathbb{R}^N$ and if $g \in C^1(\mathbb{R})$, then the following properties hold.*

(i) *If $g(0) < 0$ and if there exists $a < 0$ such that $g(a) = 0$, then there exists a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (3.3.1), with $a \le u \le 0$.*
(ii) *If $g(0) > 0$ and if there exists $b > 0$ such that $g(b) = 0$, then there exists a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (3.3.1), with $0 \le u \le b$.*

PROOF. Suppose for example $g(0) < 0$, the other case being similar. Then $\underline{u} \equiv a$ is clearly a subsolution and $\overline{u} \equiv 0$ is clearly a supersolution. □

### 3.4. The equation $-\triangle u = \lambda g(u)$

In this section, we consider a function $g \in C^1(\mathbb{R})$ and we study the problem

$$\begin{cases} -\triangle u = \lambda g(u) & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.4.1}$$

where $\lambda$ is a nonnegative parameter. We will apply Theorem 3.3.3 in order to determine the set of $\lambda$'s such that (3.4.1) has a solution in an appropriate sense.

We observe that we may assume $g(0) \neq 0$, since otherwise there is always the trivial solution $u = 0$. Next, by possibly changing $g$ to $-g$, we may assume $g(0) > 0$. Finally, if $g(b) = 0$ for some $b > 0$, then the existence of a solution for every $\lambda > 0$ follows from Corollary 3.3.6 (ii). Therefore, we need only consider the case $g(u) > 0$ for all $u \geq 0$. Our first result is the following.

Theorem 3.4.1. *Let $\Omega$ be a bounded, connected, open subset of $\mathbb{R}^N$. Let $g \in C^1(\mathbb{R})$ and assume $g(u) > 0$ for all $u \geq 0$. There exists $0 < \lambda^* \leq \infty$ with the following properties.*

(i) *For every $\lambda \in [0, \lambda^*)$, there exists a (unique) minimal solution $u_\lambda \geq 0$, $u_\lambda \in H^1_0(\Omega) \cap L^\infty(\Omega)$ of (3.4.1). $u_\lambda$ is minimal in the sense that any supersolution $v \geq 0$ of (3.4.1) satisfies $v \geq u_\lambda$. In addition, $u_\lambda \in C(\Omega)$.*
(ii) *The map $u \mapsto u_\lambda$ is increasing $(0, \infty) \to H^1_0(\Omega) \cap L^\infty(\Omega)$.*
(iii) *If $\lambda^* < \infty$ and if $\lambda > \lambda^*$, then there is no solution $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ of (3.4.1).*

Proof. The proof relies on the results of Sections 3.1 and 3.3. We proceed in three steps.

Step 1. If $\lambda > 0$ is sufficiently small, then the equation (3.4.1) has a minimal solution $u_\lambda \geq 0$, $u_\lambda \in H^1_0(\Omega) \cap L^\infty(\Omega)$. We first note that, since $g(0) > 0$, $0$ is a subsolution of (3.4.1) for all $\lambda \geq 0$. Next, let $R > 0$ be sufficiently large so that $\Omega \subset B(0, R)$ and set $w(x) = \cosh R - \cosh x_1$. It follows that $w > 0$ on $\Omega$, $w \in C^\infty(\overline{\Omega}) \subset H^1(\Omega)$ and $-\Delta w = \cosh x_1 \geq 1$. Therefore, if $\lambda > 0$ is sufficiently small so that $1/\lambda \geq \sup\{g(s);\, 0 \leq s \leq \cosh R\}$, then $-\Delta w \geq \lambda g(w)$, so that $w$ is a supersolution of (3.4.1). The result now follows from Theorem 3.3.3.

Step 2. Construction of $\lambda^*$ and proof of (i) and (iii). We set

$$\Lambda = \{\lambda \geq 0;\ (3.4.1) \text{ has a nonnegative solution in } H^1_0(\Omega) \cap L^\infty(\Omega)\}, \tag{3.4.2}$$

and

$$\lambda^* = \sup \Lambda. \tag{3.4.3}$$

It follows from Step 1 that $\Lambda \neq \emptyset$, so that $0 < \lambda^* \leq \infty$. Consider now $0 \leq \lambda < \lambda^*$. It follows from (3.4.3) that there exists $\overline{\lambda} \geq \lambda$ such that $\overline{\lambda} \in \Lambda$. Let $\overline{u} \in H^1_0(\Omega) \cap L^\infty(\Omega)$, $\overline{u} \geq 0$ be a solution of (3.4.1) with $\overline{\lambda}$. It follows that $-\Delta\overline{u} = \overline{\lambda} g(\overline{u}) \geq \lambda g(\overline{u})$, so that $\overline{u}$ is a supersolution of (3.4.1). Since, as observed above, $0$ is a subsolution of (3.4.1), it follows from Theorem 3.3.3 that the equation (3.4.1) has a minimal solution $u_\lambda \geq 0$, $u_\lambda \in H^1_0(\Omega) \cap L^\infty(\Omega)$. In addition, since $-\Delta u_\lambda = \lambda g(u_\lambda) \in L^\infty(\Omega)$, we see that $u_\lambda \in C(\Omega)$ by Corollary 4.2.7. This proves part (i), and part (iii) follows from (3.4.2)-(3.4.3).

Step 3. Proof of (ii). Let $0 \leq \lambda < \mu < \lambda^*$. It is clear that $u_\mu$ is a supersolution of (3.4.1). Since $u_\mu \geq 0$, we deduce from part (i) that $u_\mu \geq u_\lambda$. On the other hand, since $g > 0$ and $\lambda > \mu$, it is clear that $u_\mu \not\equiv u_\lambda$. Let

$$\gamma = \max\{|g'(s)|;\, 0 \leq s \leq \|u_\mu\|_{L^\infty}\}.$$

Setting $w = u_\mu - u_\lambda$, it follows that

$$\begin{aligned}-\Delta w + \lambda\gamma w &= \lambda\gamma w + \mu g(u_\mu) - \lambda g(u_\lambda)\\ &\geq \lambda[\gamma(u_\mu - u_\lambda) + g(u_\mu) - g(u_\lambda)] \geq 0.\end{aligned}$$

By the strong maximum principle (Theorem 3.1.2), we deduce that $w > 0$, i.e. $u_\mu > u_\lambda$ in $\Omega$. This completes the proof. □

REMARK 3.4.2. Here are some comments on Theorem 3.4.1.

(i) The mapping $\lambda \mapsto u_\lambda$ can have discontinuities, even for a smooth nonlinearity $g$. On the other hand, if $g$ is convex (or concave) on $[0, \infty)$, then the mapping $\lambda \mapsto u_\lambda$ is smooth. On these questions, see for example [**18**].

(ii) The conclusion of Theorem 3.4.1 can be strengthened. In particular, for $\lambda = \lambda^*$, there always exists a solution of (3.4.1) in an appropriate weak sense. That solution is unique. Moreover, if $\lambda^* < \infty$ and $\lambda > \lambda^*$, then there is no solution of (3.4.1), even in a weak sense. See in particular [**12, 36**] .

The parameter $\lambda^*$ introduced in Theorem 3.4.1 can be finite or infinite, depending on the behavior of $g(u)$ as $u \to \infty$, as shows the following result.

PROPOSITION 3.4.3. *Let $\Omega$ be a bounded, connected, open subset of $\mathbb{R}^N$. Let $g \in C^1(\mathbb{R})$ and assume $g(u) > 0$ for all $u \geq 0$. With the notation of Theorem 3.4.1, the following properties hold.*

(i) *If* $\dfrac{g(u)}{u} \underset{u\to\infty}{\longrightarrow} 0$, *then* $\lambda^* = \infty$.

(ii) *If* $\liminf\limits_{u\to\infty} \dfrac{g(u)}{u} > 0$, *then* $\lambda^* < \infty$.

PROOF. (i) By Theorem 3.3.3, we need only construct a nonnegative, bounded supersolution of (3.4.1) for every $\lambda > 0$. (Recall that 0 is always a subsolution.) To do this, we consider the function $w(x) = \cosh R - \cosh x_1$ introduced in the Step 1 of the proof of Theorem 3.4.1. Fix $\lambda > 0$ and let $k > 0$ to be chosen later. If $\overline{u} = kw$, then $-\Delta\overline{u} = k \cosh x_1 \geq k$. Since $w \leq \cosh R$, we deduce that

$$-\Delta\overline{u} \geq \frac{k}{2} + \frac{k}{2} \geq \frac{kw}{2\cosh R} + \frac{k}{2} = \frac{1}{2\cosh R}\overline{u} + \frac{k}{2}. \tag{3.4.4}$$

Since $g(u)/u \to 0$ as $u \to \infty$, we see that for every $\varepsilon > 0$, there exists $C_\varepsilon$ such that $g(u) \leq \varepsilon u + C_\varepsilon$ for all $u \geq 0$. In particular, there exists $K$ such that

$$\lambda g(u) \leq \frac{1}{2\cosh R}u + K,$$

for all $u \geq 0$. Applying (3.4.4), we deduce that

$$-\Delta\overline{u} \geq \lambda g(\overline{u}) + \frac{k}{2} - K.$$

Thus if we choose $k \geq 2K$, then $\overline{u}$ is a supersolution of (3.4.1). This proves part (i).

(ii) Let $\varphi_1 > 0$ be the first eigenfunction of $-\Delta$ (see Section 3.2). It follows from (3.4.1) that

$$\lambda_1 \int_\Omega u_\lambda\varphi_1 = \lambda\int_\Omega g(u_\lambda)\varphi_1, \tag{3.4.5}$$

for all $0 \leq \lambda < \lambda^*$. By assumption, there exist $\eta, K > 0$ such that $g(u) \geq \eta u - K$ for all $u \geq 0$; and so,

$$\lambda\int_\Omega u_\lambda\varphi_1 \leq \frac{\lambda_1}{\eta}\int_\Omega g(u_\lambda)\varphi_1 + \frac{K\lambda_1}{\eta}\|\varphi_1\|_{L^1}. \tag{3.4.6}$$

On the other hand, $g$ is clearly bounded from below on $[0,\infty)$, i.e. there exists $\varepsilon > 0$ such that $g(u) \geq \varepsilon$ for all $u > 0$. It then follows from (3.4.5) and (3.4.6) that

$$\Big(\lambda - \frac{\lambda_1}{\eta}\Big)\varepsilon\|\varphi\|_{L^1} \leq \Big(\lambda - \frac{\lambda_1}{\eta}\Big)\int_\Omega g(u_\lambda)\varphi_1 \leq \frac{K\lambda_1}{\eta}\|\varphi_1\|_{L^1},$$

for $0 \leq \lambda < \lambda^*$, which implies $\lambda^* \leq (K+\varepsilon)\lambda_1/\eta\varepsilon < \infty$. □

Assuming $\Omega$ is sufficiently smooth, we study the "stability" of the solutions $u_\lambda$.

PROPOSITION 3.4.4. *Let $\Omega$ be a bounded, connected, open subset of $\mathbb{R}^N$. Let $g \in C^1(\mathbb{R})$ and assume $g(u) > 0$ for all $u \geq 0$. If $\partial\Omega$ is of class $C^2$ then, with the notation of Theorem 3.4.1, the following properties hold.*

(i) *$\lambda_1(-\Delta - \lambda g'(u_\lambda)) \geq 0$ for every $0 \leq \lambda < \lambda^*$, where $\lambda_1$ is defined by (2.1.5).*
(ii) *If $g$ is convex or concave on $[0,\infty)$, then $\lambda_1(-\Delta - \lambda g'(u_\lambda)) > 0$ for every $0 \leq \lambda < \lambda^*$.*

PROOF. Let $\lambda_1 = \lambda_1(-\Delta - \lambda g'(u_\lambda))$ be as defined by (2.1.5) and let $\varphi_1 > 0$ be the corresponding first eigenvector (see Section 3.2). It follows that

$$-\Delta\varphi_1 = \lambda g'(u_\lambda)\varphi_1 + \lambda_1\varphi_1. \tag{3.4.7}$$

We observe that, by Theorem 3.1.4 and Remark 3.1.5 (for the lower estimate) and Theorem 4.3.1 and Remark 4.3.2 (i) (for the upper estimate), there exist $0 < k < K$ such that

$$k\mathrm{d}(x,\partial\Omega) \leq \varphi_1(x) \leq K\mathrm{d}(x,\partial\Omega), \tag{3.4.8}$$

for all $x \in \Omega$. As well, it follows that for every $0 \leq \lambda < \lambda^*$ there exist $0 < c_\lambda < C_\lambda$ such that

$$c_\lambda\mathrm{d}(x,\partial\Omega) \leq u_\lambda(x) \leq C_\lambda\mathrm{d}(x,\partial\Omega), \tag{3.4.9}$$

for all $x \in \Omega$. We now proceed in two steps.

STEP 1. Proof of (i). Assume by contradiction that $\lambda_1(-\Delta - \lambda g'(u_\lambda)) < 0$. Let $\varepsilon > 0$ and let $w_\varepsilon = u_\lambda - \varepsilon\varphi_1$. It follows from (3.4.1) and (3.4.7) that

$$\begin{aligned} -\Delta w_\varepsilon - \lambda g(w_\varepsilon) = -\lambda(g(u_\lambda - \varepsilon\varphi_1) - g(u_\lambda) \\ + \varepsilon\varphi_1[g'(u_\lambda) + (\lambda_1/\lambda)]). \end{aligned} \tag{3.4.10}$$

On the other hand, since $g$ is $C^1$ and $u_\lambda, \varphi_1 \in L^\infty(\Omega)$, we see that

$$g(u_\lambda - \varepsilon\varphi_1) - g(u_\lambda) + \varepsilon\varphi_1 g'(u_\lambda) = o(\varepsilon\varphi_1). \tag{3.4.11}$$

Since $\lambda_1 < 0$, we deduce from (3.4.10)-(3.4.11) that $-\Delta w_\varepsilon \geq \lambda g(w_\varepsilon)$ for all sufficiently small $\varepsilon > 0$, which implies that $w_\varepsilon$ is a supersolution of (3.4.1). Note that 0 is a subsolution and that by (3.4.8)-(3.4.9) , $0 \leq w_\varepsilon < u_\lambda$ if $\varepsilon > 0$ is sufficiently small. It then follows from Theorem 3.3.3 that there exists a solution $0 \leq w \leq w_\varepsilon < u_\lambda$ of (3.4.1), which contradicts the minimality of $u_\lambda$.

STEP 2. Proof of (ii). The result is clear if $\lambda = 0$, so we consider $\lambda > 0$. Assume by contradiction that $\lambda_1(-\Delta - \lambda g'(u_\lambda)) = 0$ and let $0 < \mu < \lambda^*$ to be chosen later. It then follows from (3.4.7) and (3.4.1) (for $\lambda$ and $\mu$) that

$$\int_\Omega \nabla\varphi_1\cdot\nabla u_\lambda = \lambda\int_\Omega g'(u_\lambda)\varphi_1 u_\lambda, \quad \int_\Omega \nabla\varphi_1\cdot\nabla u_\mu = \lambda\int_\Omega g'(u_\lambda)\varphi_1 u_\mu,$$

and

$$\int_\Omega \nabla\varphi_1\cdot\nabla u_\lambda = \lambda\int_\Omega g(u_\lambda)\varphi_1, \quad \int_\Omega \nabla\varphi_1\cdot\nabla u_\mu = \mu\int_\Omega g(u_\mu)\varphi_1.$$

It follows that

$$\lambda\int_\Omega g'(u_\lambda)\varphi_1 u_\lambda = \lambda\int_\Omega g(u_\lambda)\varphi_1, \quad \lambda\int_\Omega g'(u_\lambda)\varphi_1 u_\mu = \mu\int_\Omega g(u_\mu)\varphi_1,$$

from which we deduce that

$$\int_\Omega [g(u_\mu) - g(u_\lambda) - (u_\mu - u_\lambda)g'(u_\lambda)]\varphi_1 = \frac{\lambda - \mu}{\lambda} \int_\Omega g(u_\mu)\varphi_1. \tag{3.4.12}$$

Next, we observe that by (3.4.8),

$$\int_\Omega g(u_\mu)\varphi_1 > 0. \tag{3.4.13}$$

We now argue as follows. If $g$ is convex, then the integrand in the left-hand side of (3.4.12) is nonnegative. We then chose $\mu > \lambda$ so that by (3.4.13) the right-hand side of (3.4.12) is negative, yielding a contradiction. If $g$ is concave, then the integrand in the left-hand side of (3.4.12) is nonpositive and we obtain a contradiction by choosing $\lambda < \mu$. □

REMARK 3.4.5. We give an explicit characterization of all solutions of (3.4.1) in a model case. Let $\Omega = (0, \ell)$ and $g(u) = e^u$, i.e. consider the problem

$$-u'' = \lambda e^u, \quad u(0) = u(\ell) = 0. \tag{3.4.14}$$

Note first that by any solution of (3.4.14) is positive in $\Omega$. Applying Theorem 1.2.3, we see that there exists a solution of (3.4.14) every time there exists $x > 0$ such that

$$\int_0^x \frac{ds}{\sqrt{e^x - e^s}} = \ell\sqrt{\lambda/2}. \tag{3.4.15}$$

Let $H(x)$ be the left-hand side of (3.4.15). Setting successively $\sigma = e^s$, $\tau = e^x - \sigma$ and $\theta = e^{-x}\tau$, we find

$$\begin{aligned} H(x) &= \int_1^{e^x} \frac{d\sigma}{\sigma\sqrt{e^x - \sigma}} = \int_0^{e^x - 1} \frac{d\tau}{(e^x - \tau)\sqrt{\tau}} \\ &= \frac{1}{\sqrt{e^x}} \int_0^{1 - e^{-x}} \frac{d\theta}{(1 - \theta)\sqrt{\theta}}. \end{aligned}$$

Since

$$\frac{1}{(1-\theta)\sqrt{\theta}} = \frac{d}{d\theta} \log \frac{1 + \sqrt{\theta}}{1 - \sqrt{\theta}},$$

we deduce that

$$H(x) = F(1 - e^{-x}),$$

where

$$F(y) = \sqrt{1-y} \log \frac{1 + \sqrt{y}}{1 - \sqrt{y}},$$

for $y \in [0, 1)$. We note that $F(0) = 0 = \lim_{y \uparrow 1} F(y)$, and that

$$F'(y) = \frac{1}{2\sqrt{y}\sqrt{1-y}} \Big( 2 - \sqrt{y} \log \frac{1 + \sqrt{y}}{1 - \sqrt{y}} \Big).$$

Since the function $y \mapsto \sqrt{y} \log \dfrac{1 + \sqrt{y}}{1 - \sqrt{y}}$ is increasing, there exists a unique $y_0 \in (0, 1)$ such that

$$\sqrt{y_0} \log \frac{1 + \sqrt{y_0}}{1 - \sqrt{y_0}} = 2.$$

Thus $F$ is increasing on $(0, y_0)$ and decreasing on $(y_0, 1)$. If

$$\lambda^* = \frac{2}{\ell^2} \Big( \max_{y \in (0,1)} F(y) \Big)^2,$$

then it follows that if $0 < \lambda < \lambda^*$, then there exist exactly two solutions of (3.4.14); if $\lambda = \lambda^*$, then there exists exactly one solution of (3.4.14); if $\lambda > \lambda^*$, then there is no solution of (3.4.14).

CHAPTER 4

# Regularity and qualitative properties

In this section, we study the regularity and symmetry properties of the solutions of nonlinear elliptic equations. We begin by studying the regularity for linear equations, then use bootstrap arguments in the nonlinear case. For the symmetry properties, we use the "moving planes" technique, based on the maximum principle.

## 4.1. Interior regularity for linear equations

In this section, we study the interior regularity of the solutions of the equation

$$-\Delta u + \lambda u = f, \tag{4.1.1}$$

in $\mathcal{D}'(\Omega)$. Our first result concerns the case where the equation holds in the whole space $\mathbb{R}^N$.

PROPOSITION 4.1.1. *Let $m \in \mathbb{Z}$ and $1 < p < \infty$. Suppose $\lambda > 0$ and let $v, h \in \mathcal{S}'(\mathbb{R}^N)$ satisfy (4.1.1) in $\mathcal{S}'(\mathbb{R}^N)$. If $h \in W^{m,p}(\mathbb{R}^N)$, then $v \in W^{m+2,p}(\mathbb{R}^N)$, and there exists a constant $C$ such that $\|v\|_{W^{m+2,p}} \le C\|h\|_{W^{m,p}}$.*

PROOF. Taking the Fourier transform of (4.1.1), we obtain $(\lambda + 4\pi^2|\xi|^2)\widehat{v} = \widehat{h}$ in $\mathcal{S}'(\mathbb{R}^N)$, so that $(\lambda + 4\pi^2|\xi|^2)^{\frac{m+2}{2}}\widehat{v} = (\lambda + 4\pi^2|\xi|^2)^{\frac{m}{2}}\widehat{f}$, and the result follows from Theorem 5.2.3. ☐

We now consider the case of a general domain $\Omega$.

PROPOSITION 4.1.2. *Let $\lambda \in \mathbb{R}$ and let $u, f \in \mathcal{D}'(\Omega)$ satisfy the equation (4.1.1) in $\mathcal{D}'(\Omega)$.*

(i) *If $f \in W^{m,p}_{\text{loc}}(\Omega)$ and $u \in W^{n,p}_{\text{loc}}(\Omega)$ for some $m \ge 0$, $n \in \mathbb{Z}$ and $1 < p < \infty$, then $u \in W^{m+2,p}_{\text{loc}}(\Omega)$. In addition, for every $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, there exists a constant $C$ (depending only on $m$, $\Omega'$ and $\Omega''$) such that $\|u\|_{W^{m+2,p}(\Omega'')} \le C(\|f\|_{W^{m,p}(\Omega')} + \|u\|_{W^{n,p}(\Omega')})$.*

(ii) *If $f \in C^\infty(\Omega)$ and $u \in W^{n,p}_{\text{loc}}(\Omega)$ for some $n \in \mathbb{Z}$ and $1 < p < \infty$, then $u \in C^\infty(\Omega)$.*

PROOF. We proceed in two steps.

STEP 1. Consider $\omega'' \subset\subset \omega' \subset\subset \Omega$, $k \in \mathbb{Z}$ and $1 < p < \infty$. If $u \in W^{k,p}(\omega')$ and $f \in W^{k-1,p}(\omega')$ solve the equation (4.1.1) in $\mathcal{D}'(\omega')$, then $u \in W^{k+1,p}(\omega'')$ and there exists $C$ such that $\|u\|_{W^{k+1,p}(\omega'')} \le C(\|f\|_{W^{k-1,p}(\omega')} + \|u\|_{W^{k,p}(\omega')})$. To show this, consider $\rho \in C^\infty_{\text{c}}(\mathbb{R}^N)$ such that $\rho \equiv 1$ on $\omega''$ and $\operatorname{supp} \rho \subset \omega'$ and define $v \in \mathcal{D}'(\mathbb{R}^N)$ by $v = \rho u$, i.e.

$$(v, \varphi)_{\mathcal{D}'(\mathbb{R}^N), \mathcal{D}(\mathbb{R}^N)} = (u, \rho\varphi)_{\mathcal{D}'(\omega'), \mathcal{D}(\omega')}.$$

It is not difficult to show that $v \in W^{k,p}(\mathbb{R}^N)$ and that $\|v\|_{W^{k,p}(\mathbb{R}^N)} \le C\|u\|_{W^{k,p}(\omega')}$. An easy calculation shows that $v$ solves the equation

$$-\triangle v + v = T_1 + T_2 + T_3, \tag{4.1.2}$$

in $\mathcal{D}'(\mathbb{R}^N)$, where the distributions $T_1$, $T_2$ and $T_3$ are defined by

$$(T_1,\varphi)_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = (f+(1-\lambda)u,\rho\varphi)_{\mathcal{D}'(\omega'),\mathcal{D}(\omega')},$$
$$(T_2,\varphi)_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = -(u,\varphi\triangle\rho)_{\mathcal{D}'(\omega'),\mathcal{D}(\omega')},$$
$$(T_3,\varphi)_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = -2(\nabla u,\varphi\nabla\rho)_{\mathcal{D}'(\omega'),\mathcal{D}(\omega')},$$

for every $\varphi \in C^\infty_c(\mathbb{R}^N)$. It follows easily that $T_j \in W^{k-1,p}(\mathbb{R}^N)$ and that

$$\|T_j\|_{W^{k-1,p}(\mathbb{R}^N)} \le C(\|f\|_{W^{k-1,p}(\omega')} + \|u\|_{W^{k,p}(\omega')}),$$

for $j=1,2,3$. Applying (4.1.2) and Proposition 4.1.1, we deduce $v \in W^{k+1,p}(\mathbb{R}^N)$ and $\|v\|_{W^{k+1,p}(\mathbb{R}^N)} \le C(\|f\|_{W^{k-1,p}(\omega')} + \|u\|_{W^{k,p}(\omega')})$. The result follows, since the restrictions of $u$ and $v$ to $\omega''$ coincide.

Step 2. Conclusion. Without loss of generality, we may assume $n=-\ell\le 0$. Let $\Omega'' \subset\subset \Omega' \subset\subset \Omega$. Consider now a family $(\omega_j)_{0\le j\le m+\ell+1}$ of open subsets of $\Omega$, such that

$$\Omega'' = \omega_{m+\ell+1} \subset\subset \cdots \subset\subset \omega_0 \subset\subset \Omega'$$

(one constructs easily such a family). It follows from Step 1 that $u \in W^{-\ell+1,p}(\omega_0)$ and that

$$\begin{aligned}\|u\|_{W^{-\ell+1,p}(\omega_0)} &\le C(\|f\|_{W^{-\ell-1,p}(\Omega')} + \|u\|_{W^{-\ell,p}(\Omega')})\\ &\le C(\|f\|_{W^{m,p}(\Omega')} + \|u\|_{W^{n,p}(\Omega')}).\end{aligned} \tag{4.1.3}$$

We deduce from (4.1.3) and Proposition 4.1.1 that $u \in W^{-\ell+2,p}(\omega_1)$ and

$$\begin{aligned}\|u\|_{W^{-\ell+2,p}(\omega_1)} &\le C(\|f\|_{W^{-\ell,p}(\omega_0)} + \|u\|_{W^{-\ell+1,p}(\omega_0)})\\ &\le C(\|f\|_{W^{m,p}(\Omega')} + \|u\|_{W^{n,p}(\Omega')}).\end{aligned}$$

Iterating the above argument, one shows that $u \in W^{m+2,p}(\omega_{m+\ell+1}) = W^{m+2,p}(\Omega'')$ and that there exists $C$ such that $\|u\|_{W^{m+2,p}(\Omega'')} \le C(\|f\|_{W^{m,p}(\Omega')} + \|u\|_{W^{n,p}(\Omega')})$. Hence property (i), since $\Omega'$ and $\Omega''$ are arbitrary. Property (ii) follows from Property (i) and the fact that $C^\infty(\Omega) = \cap_{m\ge 0} W^{m,p}_{\mathrm{loc}}(\Omega)$ (see Corollary 5.4.17). □

## 4.2. $L^p$ regularity for linear equations

In this section, we consider an open subset $\Omega \subset \mathbb{R}^N$ and we study the $L^p$ regularity for solutions of the linear equation

$$\begin{cases} -\triangle u + \lambda u = f & \text{in} \quad \Omega,\\ u = 0 \quad \text{in} \quad \partial\Omega. \end{cases} \tag{4.2.1}$$

It follows from Theorem 2.1.4 that if $\lambda > -\lambda_1$ with $\lambda_1 = \lambda_1(-\Delta)$ defined by (2.1.5), then for every $f \in H^{-1}(\Omega)$, the equation (4.2.1) has a unique weak solution $u \in H^1_0(\Omega)$. We begin with the following result.

Theorem 4.2.1. *Let $\lambda > 0$, let $f \in H^{-1}(\Omega)$ and let $u \in H^1_0(\Omega)$ be the solution of (4.2.1). If $f \in L^p(\Omega)$ for some $p \in [1,\infty]$, then $u \in L^p(\Omega)$ and $\lambda\|u\|_{L^p} \le \|f\|_{L^p}$.*

Proof. Let $\varphi \in C^1(\mathbb{R},\mathbb{R})$. Assume that $\varphi(0)=0$, $\varphi' \ge 0$ and $\varphi' \in L^\infty(\mathbb{R})$. It follows from Proposition 5.3.1 that $\varphi(u) \in H^1_0(\Omega)$ and that $\nabla\varphi(u) = \varphi'(u)\nabla u$ a.e. in $\Omega$. Therefore, taking the $H^{-1} - H^1_0$ duality product of (4.2.1) with $\varphi(u)$, we obtain

$$\int_\Omega \varphi'(u)|\nabla u|^2\,dx + \lambda\int_\Omega u\varphi(u)\,dx = \int_\Omega f\varphi(u)\,dx;$$

and so

$$\lambda\int_\Omega u\varphi(u)\,dx \le \int_\Omega f\varphi(u)\,dx.$$

Assume that $|\varphi(u)| \le |u|^{p-1}$. It follows that $|\varphi(u)| \le (u\varphi(u))^{\frac{p-1}{p}}$; and so

$$\lambda \int_\Omega u\varphi(u)\,dx \le \|f\|_{L^p} \Big( \int_\Omega u\varphi(u)\,dx \Big)^{\frac{p-1}{p}}.$$

We deduce that

$$\lambda \Big( \int_\Omega u\varphi(u)\,dx \Big)^{\frac{1}{p}} \le \|f\|_{L^p}, \tag{4.2.2}$$

and we consider separately two cases.

Case 1: $p \le 2$ Given $\varepsilon > 0$, let $\varphi(u) = u(\varepsilon + u^2)^{\frac{p-2}{2}}$. It follows from (4.2.2) that

$$\lambda \Big( \int_\Omega u^2 (\varepsilon + u^2)^{\frac{p-2}{2}}\,dx \Big)^{\frac{1}{p}} \le \|f\|_{L^p}.$$

Letting $\varepsilon \downarrow 0$ and applying Fatou's Lemma yields the desired result.

Case 2: $2 < p \le \infty$ We use a duality argument. Given $h \in C_c^\infty(\Omega)$, let $v \in H_0^1(\Omega)$ be the solution of (4.2.1) with $f$ replaced by $h$. We have

$$\begin{aligned}\int_\Omega uh &= (u, -\triangle v + \lambda v)_{H_0^1, H^{-1}} = (-\triangle u + \lambda u, v)_{H^{-1}, H_0^1} \\ &= (f, v)_{H^{-1}, H_0^1} = \int_\Omega fv.\end{aligned}$$

Therefore,

$$\Big| \int_\Omega uh \Big| \le \|f\|_{L^p} \|v\|_{L^{p'}} \le \frac{1}{\lambda} \|f\|_{L^p} \|h\|_{L^{p'}},$$

by the result of Case 1 (since $p' < 2$). Since $h \in C_c^\infty(\Omega)$ is arbitrary, we deduce that $\|u\|_{L^p} \le \lambda^{-1} \|f\|_{L^p}$. □

For some $\lambda < 0$, one can still obtain $L^\infty$ regularity results. More precisely, we have the following.

Theorem 4.2.2. *Let $\lambda_1 = \lambda_1(-\Delta)$ be defined by (2.1.5) and let $\lambda > -\lambda_1$. Let $f \in H^{-1}(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of (4.2.1). If $f \in L^p(\Omega) + L^\infty(\Omega)$ for some $p > 1$, $p > N/2$, then $u \in L^\infty(\Omega)$. Moreover, given $1 \le r < \infty$, there exists a constant $C$ independent of $f$ such that*

$$\|u\|_{L^\infty} \le C(\|f\|_{L^p + L^\infty} + \|u\|_{L^r}).$$

*In particular, $\|u\|_{L^\infty} \le C(\|f\|_{L^p+L^\infty} + \|f\|_{H^{-1}})$.*

Proof. The proof we follow is adapted from Hartman and Stampacchia [25] (see also Brezis and Lions [13]). By homogeneity, we may assume that $\|u\|_{L^r} + \|f\|_{L^p+L^\infty} \le 1$. In particular, $f = f_1 + f_2$ with $\|f_1\|_{L^p} \le 1$ and $\|f_2\|_{L^\infty} \le 1$. Since $-u$ solves the same equation as $u$, with $f$ replaced by $-f$ (which satisfies the same assumptions), it is sufficient to estimate $\|u^+\|_{L^\infty}$. Set $T = \|u^+\|_{L^\infty} \in [0, \infty]$ and assume that $T > 0$. For $t \in (0, T)$, set $v(t) = (u - t)^+$. We have $v(t) \in H_0^1(\Omega)$ by Corollary 5.3.6. Let now $\alpha(t) = |\{x \in \Omega, u(x) > t\}|$ for all $t > 0$. Note that $\alpha(t)$ is always finite. In particular, since $v(t) \in L^2(\Omega)$ is supported in $\{x \in \Omega, u(x) > t\}$, we have $v(t) \in L^1(\Omega)$. We set

$$\beta(t) = \int_\Omega v(t)\,dx.$$

Integrating the function $1_{\{u>s\}}(x)$ on $(t, \infty) \times \Omega$ and applying Fubini's Theorem, we obtain

$$\beta(t) = \int_t^\infty \alpha(s)\,ds,$$

so that $\beta \in W^{1,1}_{\mathrm{loc}}(0, \infty)$ and

$$\beta'(t) = -\alpha(t), \tag{4.2.3}$$

for almost all $t > 0$. The idea of the proof is to obtain a differential inequality on $\beta(t)$ which implies that $\beta(t)$ must vanish for $t$ large enough. Taking the $H^{-1} - H_0^1$ duality product of (4.2.1) with $v(t)$, we obtain

$$\int_\Omega \nabla u \cdot \nabla v(t) + \lambda \int_\Omega uv(t) = (f, v(t))_{H^{-1},H_0^1},$$

for every $t > 0$. Therefore, by applying formula (5.3.3) and the property $v(t) \in L^1(\Omega)$, we deduce that

$$\int_\Omega \{|\nabla v(t)|^2 + \lambda |v(t)|^2\}\, dx = \int_\Omega (f - t\lambda) v(t)\, dx.$$

Since $\lambda > -\lambda_1$, we deduce by applying (2.1.8) that

$$\|v(t)\|_{H^1}^2 \le C \int_\Omega (f - t\lambda) v(t)\, dx \le C \int_\Omega (|f| + t|\lambda|) v(t)\, dx. \tag{4.2.4}$$

We observe that

$$\begin{aligned}\int_\Omega |f| v(t) &\le \int_\Omega (|f_1| + |f_2|) v(t) \le \|f_1\|_{L^p} \|v(t)\|_{L^{p'}} + \|f_2\|_{L^\infty} \|v(t)\|_{L^1} \\ &\le \|v(t)\|_{L^{p'}} + \|v(t)\|_{L^1},\end{aligned}$$

and we deduce from (4.2.4) that

$$\|v(t)\|_{H^1}^2 \le C(1+t)(\|v(t)\|_{L^{p'}} + \|v(t)\|_{L^1}). \tag{4.2.5}$$

Fix now $\rho > 2p'$ such that $\rho < 2N/(N-2)$ ($\rho < \infty$ if $N = 1$). (Note that this is possible since $p > \min\{1, N/2\}$.) We have in particular $H_0^1(\Omega) \hookrightarrow L^\rho(\Omega)$. Also, it follows from Hölder's inequality that $\|v(t)\|_{L^1} \le \alpha(t)^{1-\frac{1}{\rho}} \|v(t)\|_{L^\rho}$ and $\|v(t)\|_{L^{p'}} \le \alpha(t)^{\frac{1}{p'} - \frac{1}{\rho}} \|v(t)\|_{L^\rho}$. Thus we deduce from (4.2.5) that

$$\|v(t)\|_{L^\rho}^2 \le C(1+t)(\alpha(t)^{\frac{1}{p'} - \frac{1}{\rho}} + \alpha(t)^{1-\frac{1}{\rho}}) \|v(t)\|_{L^\rho}.$$

Finally, since $\beta(t) = \|v(t)\|_{L^1} \le \alpha(t)^{1-\frac{1}{\rho}} \|v(t)\|_{L^\rho}$, we obtain

$$\beta(t) \le C(1+t)(\alpha(t)^{1+\frac{1}{p'} - \frac{2}{\rho}} + \alpha(t)^{2-\frac{2}{\rho}}),$$

which we can write as

$$\beta(t) \le C(1+t) F(\alpha(t)),$$

with $F(s) = s^{1+\frac{1}{p'}-\frac{2}{\rho}} + s^{2-\frac{2}{\rho}}$. It follows that

$$-\alpha(t) + F^{-1}\Big(\frac{\beta(t)}{C(1+t)}\Big) \le 0. \tag{4.2.6}$$

Setting $z(t) = \dfrac{\beta(t)}{C(1+t)}$, we deduce from (4.2.3) and (4.2.6) that

$$z' + \frac{\psi(z(t))}{C(1+t)} \le 0,$$

with $\psi(s) = F^{-1}(s) + Cs$. Integrating the above differential inequality yields

$$\int_s^t \frac{d\sigma}{C(1+\sigma)} \le \int_{z(t)}^{z(s)} \frac{d\sigma}{\psi(\sigma)},$$

for all $0 < s < t < T$. If $T \le 1$, then by definition $\|u^+\|_{L^\infty} \le 1$. Otherwise, we obtain

$$\int_1^t \frac{d\sigma}{C(1+\sigma)} \le \int_{z(t)}^{z(1)} \frac{d\sigma}{\psi(\sigma)},$$

for all $1 < t < T$, which implies in particular that

$$\int_1^T \frac{d\sigma}{C(1+\sigma)} \le \int_0^{z(1)} \frac{d\sigma}{\psi(\sigma)}.$$

Note that $F(s) \approx s^{1+\frac{1}{p'}-\frac{2}{\rho}}$ as $s \downarrow 0$ and $1 + 1/p' - 2/\rho > 1$, so that $1/\psi$ is integrable near zero. Since $1/(1+\sigma)$ is not integrable at $\infty$, this implies that $T = \|u^+\|_{L^\infty} < \infty$. Moreover, $\|u^+\|_{L^\infty}$ is estimated in terms of $z(1)$, and

$$z(1) = \frac{1}{C}\int_\Omega (u-1)^+ \le \frac{1}{C}\int_{\{u>1\}} u \le \frac{1}{C}\int_{\{u>1\}} u^r \le \frac{1}{C}.$$

This completes the proof. □

One can improve the $L^p$ estimates by using Sobolev's inequalities. In particular, we have the following result.

THEOREM 4.2.3. *Let $\lambda > 0$, $f \in H^{-1}(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of (4.2.1). If $f \in L^p(\Omega)$ for some $p \in (1, \infty]$, then the following properties hold.*

(i) *If $p > N/2$, then $u \in L^p(\Omega) \cap L^\infty(\Omega)$, and there exists a constant $C$ independent of $f$ such that $\|u\|_{L^r} \le C\|f\|_{L^p}$ for all $r \in [p, \infty]$.*
(ii) *If $p = N/2$ and $N \ge 3$, then $u \in L^r(\Omega)$ for all $r \in [p, \infty)$, and there exist constants $C(r)$ independent of $f$ such that $\|u\|_{L^r} \le C(r)\|f\|_{L^p}$.*
(iii) *If $1 < p < N/2$ and $N \ge 3$, then $u \in L^p(\Omega) \cap L^{\frac{Np}{N-2p}}(\Omega)$, and there exists a constant $C$ independent of $f$ such that $\|u\|_{L^r} \le C\|f\|_{L^p}$ for all $r \in [p, Np/(N-2p)]$.*

PROOF. Property (i) follows from Theorems 4.2.1 and 4.2.2 and Hölder's inequality. It remains to establish properties (ii) and (iii). Note that in this case $N \ge 3$. By density, it is sufficient to establish these properties for $f \in C_c^\infty(\Omega)$. In this case, we have $u \in L^1(\Omega) \cap L^\infty(\Omega)$ by Theorem 4.2.1. Consider an odd, increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi'$ is bounded and define

$$\psi(x) = \int_0^x \sqrt{\varphi'(s)}\, ds. \tag{4.2.7}$$

It follows that $\psi$ is odd, nondecreasing, and that $\psi'$ is bounded. By Proposition 5.3.1, $\varphi(u)$ and $\psi(u)$ both belong to $H_0^1(\Omega)$, and

$$|\nabla\psi(u)|^2 = \varphi'(u)|\nabla u|^2 = \nabla u \cdot \nabla(\varphi(u)), \tag{4.2.8}$$

a.e. Taking the $H^{-1}-H_0^1$ duality product of (4.2.1) with $\varphi(u)$, it follows from (4.2.8) that

$$\int_\Omega (|\nabla(\psi(u))|^2 + \lambda u\varphi(u))\, dx = (f, \varphi(u))_{H^{-1},H_0^1}.$$

In addition, $x\varphi(x) \ge 0$, and it follows from (4.2.7) and Cauchy-Schwarz inequality that $x\varphi(x) \ge |\psi(x)|^2$. Therefore, there exists a constant $C$ such that

$$\|\psi(u)\|_{H^1}^2 \le C(f, \varphi(u))_{H^{-1},H_0^1}.$$

We deduce that, given any $p \in [1, \infty]$,

$$\|\psi(u)\|_{H^1}^2 \le C\|f\|_{L^p}\|\varphi(u)\|_{L^{p'}}.$$

Since $H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, it follows that

$$\|\psi(u)\|_{L^{\frac{2N}{N-2}}}^2 \le C\|f\|_{L^p}\|\varphi(u)\|_{L^{p'}}. \tag{4.2.9}$$

Consider now $1 < q < \infty$ such that $(q-1)p' \ge 1$. If $q \le 2$, let $\varphi_\varepsilon(x) = x(\varepsilon + x^2)^{\frac{q-2}{2}}$. If $q > 2$, take $\varphi_\varepsilon(x) = x|x|^{q-2}(1+\varepsilon x^2)^{\frac{2-q}{2}}$. It follows that $|\varphi_\varepsilon(x)| \le C|x|^{q-1}$ and that $|\varphi_\varepsilon(x)| \to |x|^{q-1}$ as $\varepsilon \downarrow 0$. One verifies easily that $|\psi_\varepsilon(x)|^2 \le C|x|^q$ and that

$|\psi_\varepsilon(x)|^2 \to (4(q-1)/q^2)|x|^q$. Applying (4.2.9), then letting $\varepsilon \downarrow 0$ and applying the dominated convergence theorem, it follows that

$$\|u\|^q_{L^{\frac{Nq}{N-2}}} \le C\frac{q^2}{q-1}\|f\|_{L^p}\|u\|^{q-1}_{L^{(q-1)p'}}, \tag{4.2.10}$$

for all $1 < q < \infty$ such that $(q-1)p' \ge 1$. We now prove property (ii). Suppose that $N \ge 3$ and that $p = N/2$. Apply (4.2.10) with $q > N/2$. It follows that

$$\|u\|^q_{L^{\frac{Nq}{N-2}}} \le C\frac{q^2}{q-1}\|f\|_{L^{N/2}}\|u\|^{q-1}_{L^{\frac{N(q-1)}{N-2}}}. \tag{4.2.11}$$

On the other hand, it follows from Hölder's inequality and Theorem 4.2.1 that

$$\|u\|^{q-1}_{L^{\frac{N(q-1)}{N-2}}} \le \|u\|^{\frac{(2q-N)q}{2q-N+2}}_{L^{\frac{Nq}{N-2}}}\|u\|^{\frac{N-2}{2q-N+2}}_{L^{\frac{N}{2}}} \le \|u\|^{\frac{(2q-N)q}{2q-N+2}}_{L^{\frac{Nq}{N-2}}}\|f\|^{\frac{N-2}{2q-N+2}}_{L^{\frac{N}{2}}}.$$

Substitution in (4.2.11) yields

$$\|u\|_{L^{\frac{Nq}{N-2}}} \le C(q)\|f\|_{L^{\frac{N}{2}}}.$$

Property (ii) follows from the above estimate and Theorem 4.2.1, since $q$ is arbitrary. Finally, we prove property (iii). We set $q = (N-2)p/(N-2p)$. It follows in particular that $Nq/(N-2) = (q-1)p' = Np/(N-2p)$, so that by (4.2.10)

$$\|u\|_{L^{\frac{Np}{N-2p}}} \le C'\|f\|_{L^p}.$$

Property (iii) follows from the above estimate and Theorem 4.2.1. □

Corollary 4.2.4. *Let $\lambda > 0$, let $f \in H^{-1}(\Omega)$ and let $u \in H^1_0(\Omega)$ be the solution of (4.2.1). If $f \in L^1(\Omega)$, then the following properties hold.*

(i) *If $N = 1$, then $u \in L^1(\Omega)\cap L^\infty(\Omega)$, and there exists a constant $C$ independent of $f$ and $r$ such that $\|u\|_{L^r} \le C\|f\|_{L^1}$ for all $r \in [1,\infty]$.*
(ii) *If $N \ge 2$, then $u \in L^r(\Omega)$ for all $r \in [1, N/(N-2))$ and there exists a constant $C(r)$ independent of $f$ such that $\|u\|_{L^r} \le C(r)\|f\|_{L^1}$.*

Proof. If $N = 1$, then $u \in H^1_0(\Omega) \hookrightarrow L^\infty(\Omega)$. Taking the $H^{-1} - H^1_0$ duality product of (4.2.1) with $u$, we deduce easily that there exists $\mu > 0$ such that

$$\mu\|u\|^2_{H^1} \le (f,u)_{H^{-1},H^1_0} \le \|f\|_{L^1}\|u\|_{L^\infty} \le C\|f\|_{L^1}\|u\|_{H^1}.$$

Therefore, $\mu\|u\|_{H^1} \le C\|f\|_{L^1}$, and (i) follows. In the case $N \ge 2$, we use a duality argument. Let $u$ and $f$ be as in the statement of the theorem. It follows from Theorem 4.2.1 that $u \in L^1(\Omega)$ and

$$\|u\|_{L^1} \le C\|f\|_{L^1}. \tag{4.2.12}$$

Fix $q > N/2$. Let $h \in C^\infty_{\mathrm{c}}(\Omega)$, and let $\varphi \in H^1_0(\Omega)$ be the solution of the equation $-\triangle\varphi + \lambda\varphi = h$. It follows from Theorem 4.2.3 that

$$\|\varphi\|_{L^\infty} \le C\|h\|_{L^q}. \tag{4.2.13}$$

Since

$$\begin{aligned}(f,\varphi)_{H^{-1},H^1_0} &= (-\triangle u + \lambda u, \varphi)_{H^{-1},H^1_0}\\ &= (u, -\triangle\varphi + \lambda\varphi)_{H^1_0,H^{-1}} = (u,h)_{H^1_0,H^{-1}},\end{aligned}$$

we deduce from (4.2.12)-(4.2.13) that

$$\Big|\int_\Omega uh\Big| \le \|f\|_{L^1}\|\varphi\|_{L^\infty} \le C\|f\|_{L^1}\|h\|_{L^q}.$$

Since $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$ is arbitrary, we obtain $\|u\|_{L^{q'}} \le C\|f\|_{L^1}$. Since $N/2 < q \le \infty$ is arbitrary, $1 \le q' < N/(N-2)$ is arbitrary and the result follows. □

REMARK 4.2.5. Note that the estimates of Theorem 4.2.3 and Corollary 4.2.4 are optimal in the following sense.

(i) If $N \geq 2$ and $f \in L^{\frac{N}{2}}(\Omega)$, then $u$ is not necessarily in $L^\infty(\Omega)$. For example, let $\Omega$ be the unit ball, and let $u(x) = (-\log|x|)^\gamma$ with $\gamma > 0$. Then $u \notin L^\infty(\Omega)$. On the other hand, one verifies easily that if $0 < \gamma < 1/2$ in the case $N = 2$ and $0 < \gamma < 1 - 2/N$ in the case $N \geq 3$, then $u \in H_0^1(\Omega)$ and $-\triangle u + u \in L^{\frac{N}{2}}(\Omega)$.

(ii) If $N \geq 3$ and $f \in L^1(\Omega)$, then there is no estimate of the form $\|u\|_{L^{\frac{N}{N-2}}} \leq C\|f\|_{L^1}$. (Note that since $u \in H_0^1(\Omega)$, we always have $u \in L^{\frac{N}{N-2}}(\Omega)$.) One constructs easily a counter example as follows. Let $\Omega$ be the unit ball, and let $u = z\varphi$ with $\varphi \in C_c^\infty(\Omega)$, $\varphi(0) \neq 0$, and $z(x) = |x|^{2-N}(-\log|x|)^\gamma$ with $\gamma < 0$. Then $-\triangle u + u \in L^1(\Omega)$ and $u \notin L^{\frac{N}{N-2}}(\Omega)$. By approximating $u$ by smooth functions, one deduces that there is no estimate of the form $\|u\|_{L^{\frac{N}{N-2}}} \leq C\|f\|_{L^1}$.

(iii) If $N \geq 3$ and $1 < p < N/2$, then by arguing as above one shows the following properties. There is no estimate of the form $\|u\|_{L^q} \leq C\|f\|_{L^1}$ for $q > Np/(N-2p)$. Moreover, if $f \in L^p(\Omega)$, then in general $u \notin L^q(\Omega)$ for $q > Np/(N-2p)$, $q > 2N/(N-2)$.

REMARK 4.2.6. Under some smoothness assumptions on $\Omega$, one can establish higher order $L^p$ estimates. However, the proof of these estimates is considerably more delicate. In particular, one has the following results.

(i) If $\Omega$ has a bounded boundary of class $C^2$ (in fact, $C^{1,1}$ is enough) and if $1 < p < \infty$, then one can show that for every $\lambda > 0$ and $f \in L^p(\Omega)$, there exists a unique solution $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ of equation (4.2.1), and that

$$\|u\|_{W^{2,p}} \leq C(\|u\|_{L^p} + \|f\|_{L^p}),$$

for some constant $C$ independent of $f$ (see e.g. Theorem 9.15, p.241 in Gilbarg and Trudinger [**23**]). One shows as well that for every $f \in W^{-1,p}(\Omega)$, there exists a unique solution $u \in W_0^{1,p}(\Omega)$ of equation (4.2.1) (see Agmon, Douglis and Nirenberg [**3**]).

(ii) One has partial results in the cases $p = 1$ and $p = \infty$. In particular, if $\Omega$ is bounded and smooth enough, then for every $\lambda > 0$ and $f \in L^1(\Omega)$, there exists a unique solution $u \in W_0^{1,1}(\Omega)$, such that $\triangle u \in L^1(\Omega)$, of equation (4.2.1) (see Pazy [**39**], Theorem 3.10, p.218). It follows that $\lambda\|u\|_{L^1} \leq \|f\|_{L^1}$. In general, $u \notin W^{2,1}(\Omega)$. If $\Omega$ is bounded, it follows from Theorems 2.1.4 and 4.2.1 that for every $\lambda > 0$ and $f \in L^\infty(\Omega)$, there exists a unique solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, such that $\triangle u \in L^\infty(\Omega)$, of equation (4.2.1). In general, $u \notin W^{2,\infty}(\Omega)$, even if $\Omega$ is smooth. On the other hand, it follows from property (i) above that $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, for every $p < \infty$.

COROLLARY 4.2.7. *Let $\lambda > 0$, $f \in H^{-1}(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of (4.2.1). If $f \in L^p(\Omega)$ for some $1 \leq p < \infty$ with $p > N/2$, or if $f \in C_0(\Omega)$, then $u \in L^\infty(\Omega) \cap C(\Omega)$.*

PROOF. Let $(f_n)_{n\geq 0} \subset C_c^\infty(\Omega)$ such that $f_n \to f$ in $L^p(\Omega)$ as $n \to \infty$ (we let $p = \infty$ in the case $f \in C_0(\Omega)$), and let $(u_n)_{n\geq 0}$ be the corresponding solutions of (4.2.1). It follows from Proposition 4.1.2 (ii) that $u_n \in C(\Omega)$. Moreover, $u_n \to u$ in $L^\infty(\Omega)$ by Theorem 4.2.3 (i) (or Corollary 4.2.4 (i) in the case $p = 1 = N$), and the result follows. □

### 4.3. $C_0$ regularity for linear equations

In this section we show that if $\Omega$ satisfies certain geometric assumptions, then the solution of (4.2.1) with sufficiently smooth right-hand side is continuous at $\partial\Omega$.

THEOREM 4.3.1. *If $N \geq 2$ suppose that there exists $\rho > 0$ such that for every $x_0 \in \partial\Omega$ there exists $y(x_0) \in \mathbb{R}^N$ with $|x_0 - y(x_0)| = \rho$ and $B(y_0, \rho) \cap \Omega = \emptyset$. Let $\lambda > 0$, $f \in H^{-1}(\Omega) \cap L^\infty(\Omega)$ and let $u \in H_0^1(\Omega)$ be the solution of (4.2.1). It follows that*

$$|u(x)| \leq C\|f\|_{L^\infty} \mathrm{d}(x, \partial\Omega), \tag{4.3.1}$$

*for all $x \in \Omega$, where $C$ is independent of $f$.*

PROOF. We may assume without loss of generality that $|f| \leq 1$, so that $|u| \leq \lambda^{-1}$ by Theorem 4.2.1. We may also suppose $N \geq 2$, for the case $N = 1$ is immediate. We construct a *local barrier* at every point of $\partial\Omega$. Given $c > 0$, set

$$w(x) = \begin{cases} \frac{1}{4}(\rho^2 - |x|^2) + c\log(|x|/\rho) & \text{if} \quad N = 2, \\ \frac{1}{2N}(\rho^2 - |x|^2) + c(\rho^{2-N} - |x|^{2-N}) & \text{if} \quad N \geq 3. \end{cases} \tag{4.3.2}$$

It follows that $-\triangle w = 1$ in $\mathbb{R}^N \setminus \{0\}$. Furthermore, we see that if $c$ is large enough, then there exist $\rho_1 > \rho_0 > \rho$ such that

$$w(x) > 0 \text{ for } \rho < |x| \leq \rho_1, \quad w(x) \geq \lambda^{-1} \text{ for } \rho_0 \leq |x| \leq \rho_1. \tag{4.3.3}$$

Given $c$ as above, we observe that there exists a constant $K$ such that

$$w(x) \leq K(|x| - \rho) \text{ for } \rho \leq |x| \leq \rho_1. \tag{4.3.4}$$

Let now $\widetilde{x} \in \Omega$ such that $2\mathrm{d}(\widetilde{x}, \partial\Omega) < \rho_1 - \rho$, and let $x_0 \in \partial\Omega$ be such that $|\widetilde{x} - x_0| \leq 2\mathrm{d}(\widetilde{x}, \partial\Omega)$. Set $\widetilde{\Omega} = \{x \in \Omega;\ \rho < |x - y(x_0)| < \rho_1\}$ and $v(x) = w(x - y(x_0))$ for $x \in \widetilde{\Omega}$. We note that $|\widetilde{x} - y(x_0)| > \rho$ by the geometric assumption. Moreover,

$$|\widetilde{x} - y(x_0)| \leq |\widetilde{x} - x_0| + |x_0 - y(x_0)| < \rho_1 - \rho + \rho = \rho_1,$$

so that $\widetilde{x} \in \widetilde{\Omega}$. Next, it follows from (4.3.3)-(4.3.4) that $v > 0$ on $\widetilde{\Omega}$ and that

$$\begin{aligned} 0 \leq v(\widetilde{x}) &\leq K(|\widetilde{x} - y(x_0)| - \rho) \leq K(|\widetilde{x} - x_0| + |x_0 - y(x_0)| - \rho) \\ &= K|\widetilde{x} - x_0| \leq 2K\mathrm{d}(\widetilde{x}, \partial\Omega). \end{aligned}$$

On the other hand,

$$-\triangle(u - v) + \lambda(u - v) = f - (1 + \lambda v) \leq f - 1 \leq 0,$$

in $\widetilde{\Omega}$. We claim that

$$(u - v)^+ \in H_0^1(\widetilde{\Omega}). \tag{4.3.5}$$

It then follows from the maximum principle that $u \leq v$ in $\widetilde{\Omega}$. In particular, $u(\widetilde{x}) \leq v(\widetilde{x}) \leq 2K\mathrm{d}(\widetilde{x}, \partial\Omega)$. Changing $u$ to $-u$, one obtains as well that $-u \leq v$, so that $|u(\widetilde{x})| \leq 2K\mathrm{d}(\widetilde{x}, \partial\Omega)$ for a.a. $x \in \widetilde{\Omega}$. In particular, $|u(\widetilde{x})| \leq 2K\mathrm{d}(\widetilde{x}, \partial\Omega)$. Since $\widetilde{x}$ is arbitrary, we deduce that if $x \in \Omega$ such that $2\mathrm{d}(x, \partial\Omega) < \rho_1 - \rho$, then $|u(x)| \leq 2K\mathrm{d}(x, \partial\Omega)$. For $x \in \Omega$ such that $2\mathrm{d}(x, \partial\Omega) \geq \rho_1 - \rho$, we have $u(x) \leq \lambda^{-1} \leq 2\lambda^{-1}(\rho_1 - \rho)^{-1}\mathrm{d}(x, \partial\Omega)$, and the result follows.

It now remains to establish the claim (4.3.5). Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ satisfy $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on the set $\{|x - y(x_0)| \leq \rho_0\}$ and $\varphi \equiv 0$ on the set $\{|x - y(x_0)| \geq \rho_1\}$. We note that by (4.3.3), $u \leq \lambda^{-1} \leq v$, thus $\varphi u - v \leq u - v \leq 0$ on $\widetilde{\Omega} \cap \{|x - y(x_0)| \geq \rho_0\}$. Therefore, $(\varphi u - v)^+ = (u - v)^+ = 0$ on $\widetilde{\Omega} \cap \{|x - y(x_0)| \geq \rho_0\}$. On $\widetilde{\Omega} \cap \{|x - y(x_0)| \leq \rho_0\}$, $\varphi u - v = u - v$, so that $(u - v)^+ = (\varphi u - v)^+$ in $\widetilde{\Omega}$. Let now $(u_n)_{n\geq 0} \subset C_c^\infty(\Omega)$ satisfy $u_n \to u$ in $H^1(\Omega)$ as $n \to \infty$. It follows (see Proposition 5.3.3) that $(\varphi u_n - v)^+ \to (\varphi u - v)^+ = (u - v)^+$ in $H^1(\widetilde{\Omega})$. Thus, we need only verify that $(\varphi u_n - v)^+ \in H^1_0(\widetilde{\Omega})$. This follows from Remark 5.1.10 (i), because $\varphi u_n = 0$ and $v \geq 0$ on $\partial\widetilde{\Omega}$. □

REMARK 4.3.2. Here are some comments on Theorem 4.3.1.

(i) One verifies easily that it $\partial\Omega$ is uniformly of class $C^2$, then the geometric assumption of Theorem 4.3.1 is satisfied.

(ii) One can weaken the regularity assumption on $\Omega$ and still obtain the continuity of $u$ at $\partial\Omega$. However, one does not obtain in general the estimate (4.3.1).

(iii) Without any regularity assumption, it can happen that $u \notin C_0(\Omega)$ even for some $f \in C^\infty_c(\Omega)$. For example, let $\Omega = \mathbb{R}^N \setminus \{0\}$ with $N \geq 2$ and set $\varphi(x) = \cosh x_1$ for $x \in \Omega$. It follows that $\varphi \in C^\infty(\mathbb{R}^N)$, and $-\triangle\varphi + \varphi = 0$. Let now $\psi \in C^\infty_c(\mathbb{R}^N)$ satisfy $\psi \equiv 1$, for $|x| \leq 1$ and $\psi \equiv 0$, for $|x| \geq 2$. Set $u = \varphi\psi$, so that $u \in C^\infty_c(\mathbb{R}^N)$. It is not difficult to verify that $u \in H^1_0(\Omega)$ (see Remark 5.1.10 iii). On the other hand, $-\triangle u + u = 0$ for $|x| \leq 1$ and for $|x| \geq 2$. In particular, if we set $f = -\triangle u + u$, then $f \in C^\infty_c(\Omega)$. Finally, $u \notin C_0(\Omega)$, since $u = 1$ on $\partial\Omega$.

COROLLARY 4.3.3. *Suppose $\Omega$ satisfies the assumption of Theorem 4.3.1. Let $\lambda > 0$, $f \in H^{-1}(\Omega) \cap L^\infty(\Omega)$ and let $u \in H^1_0(\Omega)$ be the solution of (4.2.1). If $f \in L^p(\Omega)$ for some $1 \leq p < \infty$ with $p > N/2$, or if $f \in C_0(\Omega)$, then $u \in C_0(\Omega)$.*

PROOF. By Corollary 4.2.7, $u \in C(\Omega)$. Continuity at $\partial\Omega$ follows from Theorem 4.3.1. □

## 4.4. Bootstrap methods

In this section, we consider an arbitrary open domain $\Omega \subset \mathbb{R}^N$ and we study the regularity of the solutions of the equation

$$\begin{cases} -\Delta u = g(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $g$ is a given nonlinearity. We use the regularity properties of the linear equation (Sections 4.1, 4.2 and 4.3 above) and bootstarp arguments. We prove two kind of results. We establish sufficient conditions (on $g$ and $u$) so that $u \in L^\infty(\Omega)$. We also prove interior regularity, assuming $u$ is (locally) bounded and $g$ is sufficiently smooth. Our first result is the following.

THEOREM 4.4.1. *Let $g(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in $x \in \Omega$ for all $t \in \mathbb{R}$ and continuous in $t \in \mathbb{R}$ for a.a. $x \in \Omega$. Assume further that there exist $p \geq 1$ and a constant $C$ such that*

$$|g(x,t)| \leq C(|t| + |t|^p), \tag{4.4.1}$$

*for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$. Let $u \in H^1_0(\Omega)$ satisfy $g(\cdot, u(\cdot)) \in H^{-1}(\Omega)$ and assume that*

$$-\Delta u = g(\cdot, u(\cdot)), \tag{4.4.2}$$

*in $H^{-1}(\Omega)$. If $N \geq 3$, suppose that $u \in L^q(\Omega)$ for some*

$$q \geq p, \quad q > \frac{N(p-1)}{2}. \tag{4.4.3}$$

*It follows that $u \in L^\infty(\Omega) \cap C(\Omega)$.*

PROOF. If $N = 1$, then the result follows from the embedding $H^1_0(\Omega) \hookrightarrow C_0(\Omega)$. If $N = 2$, $H^1_0(\Omega) \hookrightarrow L^r(\Omega)$ for all $2 \leq r < \infty$. In particular, it follows from (4.4.1) that $g(\cdot, u) \in L^N(\Omega)$, and the result follows from Corollary 4.2.7. Therefore, we may now suppose that $N \geq 3$. We first proceed to a reduction in order to eliminate the first term in the right-hand side of (4.4.1). Let $\eta \in C^\infty_c(\mathbb{R})$ satisfy $\eta(t) = 0$

for $|t| \le 1$. Set $g_1(x,t) = \eta(t)(t + g(x,t))$ and $g_2(x,t) = (1-\eta(t))(t+g(x,t))$. It follows from (4.4.1) that

$$|g_1(x,t)| \le C \min\{1, |t|\}, \tag{4.4.4}$$
$$|g_2(x,t)| \le C|t|^p, \tag{4.4.5}$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$. It follows from (4.4.4) that $g_1(\cdot,u) \in L^2(\Omega) \cap L^\infty(\Omega)$. Therefore, if we denote by $u_1 \in H^1_0(\Omega)$ the solution of the equation

$$-\Delta u_1 + u_1 = g_1(\cdot, u), \tag{4.4.6}$$

then it follows from Corollary 4.2.7 that

$$u_1 \in L^2(\Omega) \cap L^\infty(\Omega) \cap C(\Omega). \tag{4.4.7}$$

Therefore, we need only show that $u_2 = u - u_1 \in L^\infty(\Omega) \cap C(\Omega)$. We note that by (4.4.2) and (4.4.6), $u_2 \in H^1_0(\Omega)$ satisfies the equation

$$-\Delta u_2 + u_2 = g_2(\cdot, u). \tag{4.4.8}$$

We now note that, since $u \in L^2(\Omega)$, we may assume that

$$q \ge 2, \tag{4.4.9}$$

and we proceed in three steps.

Step 1. The case $q > Np/2$. It follows from (4.4.5) that $g_2(\cdot,u) \in L^{\frac{q}{p}}(\Omega)$. Since $q/p > N/2 > 1$, we deduce from Corollary 4.2.7 that $u_2 \in L^\infty(\Omega) \cap C(\Omega)$, and the result follows by applying (4.4.7).

Step 2. The case $p < q \le Np/2$. Let

$$\theta = \frac{N}{Np - 2q} > 1, \tag{4.4.10}$$

by (4.4.3). Suppose $u \in L^r(\Omega)$ for some $q \le r < Np/2$. It follows from (4.4.5) that $g_2(\cdot,u) \in L^{\frac{r}{p}}(\Omega)$. Since $r/p \ge q/p > 1$ and $r/p < N/2$, it follows from (4.4.8) and Theorem 4.2.3 (iii) that $u_2 \in L^{\theta r}(\Omega)$. We note that by (4.4.9) and (4.4.10), $\theta r > r \ge 2$, so that $u_1 \in L^{\theta r}(\Omega)$ by (4.4.7). Therefore, $u \in L^{\theta r}(\Omega)$. Let now $k$ be an integer such that $\theta^k q \le Np/2 < \theta^{k+1} q$. Using successively $r = \theta^j q$ with $j = 0, \cdots, k$ in the argument just above, we deduce that $u \in L^{\theta^{k+1} q}(\Omega)$. Since $\theta^{k+1} q > Np/2$, the result now follows by Step 1.

Step 3. The case $p = q$ $(\le Np/2)$. It follows from (4.4.5) that $g_2(\cdot,u) \in L^1(\Omega)$. By (4.4.8) and Corollary 4.2.4 (ii), we deduce that $u_2 \in L^r(\Omega)$ for all $r \in [1, N/(N-2))$. Finally, we note that

$$\frac{N}{N-2} = \frac{N}{Np-2q} q > q,$$

by (4.4.10). In particular, $N/(N-2) > 2$, so that $u \in L^r(\Omega)$ for all $r \in [2, N/(N-2))$. Thus we are reduced to the case of Step 2. □

If $|\Omega| < \infty$, then one can weaken the assumption (4.4.1), as shows the following result.

Corollary 4.4.2. *Assume $|\Omega| < \infty$. Let $g(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in $x \in \Omega$ for all $t \in \mathbb{R}$ and continuous in $t \in \mathbb{R}$ for a.a. $x \in \Omega$. Assume further that there exist $p \ge 1$ and a constant $C$ such that*

$$|g(x,t)| \le C(1 + |t|^p), \tag{4.4.11}$$

*for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$. Let $u \in H^1_0(\Omega)$ satisfy $g(\cdot, u(\cdot)) \in H^{-1}(\Omega)$ and (4.4.2) in $H^{-1}(\Omega)$. If $N \ge 3$, assume further (4.4.3). It follows that $u \in L^\infty(\Omega) \cap C(\Omega)$.*

PROOF. The proof of Theorem 4.4.1 applies, except for the beginning which requires a minor modification. Instead of (4.4.4), the function $g_1$ satisfies $|g_1(x,t)| \le C$; and so, $g_1(\cdot,u) \in L^\infty(\Omega) \hookrightarrow L^2(\Omega)$, since $|\Omega| < \infty$. The remaining of the proof is then unchanged. □

COROLLARY 4.4.3. *Let $g(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable in $x \in \Omega$ for all $t \in \mathbb{R}$ and continuous in $t \in \mathbb{R}$ for a.a. $x \in \Omega$. Assume further (4.4.10) (or (4.4.11) if $|\Omega| < \infty$). Let $u \in H_0^1(\Omega)$ satisfy $g(\cdot,u(\cdot)) \in H^{-1}(\Omega)$ and (4.4.2) in $H^{-1}(\Omega)$. If $N \ge 3$, assume further that*

$$p < \frac{N+2}{N-2}. \tag{4.4.12}$$

*It follows that $u \in L^\infty(\Omega) \cap C(\Omega)$.*

PROOF. If $N \ge 3$, then, since $u \in H_0^1(\Omega)$, we have $u \in L^q(\Omega)$ with $q = 2N/(N-2)$. We deduce from (4.4.12) that $q$ satisfies (4.4.3). The result now follows from Theorem 4.4.1 (or Corollary 4.4.2 if $|\Omega| < \infty$). □

REMARK 4.4.4. Under the assumptions of Theorem 4.4.1 (or those of Corollary 4.4.2 if $|\Omega| < \infty$), we have $g(\cdot,u) \in L^2(\Omega) \cap L^\infty(\Omega)$. Therefore, if $\Omega$ has a bounded boundary of class $C^2$ (or, more generally, if $\Omega$ satisfies the assumptions of Theorem 4.3.1), then it follows from Corollary 4.3.3 and Remark 4.3.2 (i) that $u \in C_0(\Omega)$.

We now study higher order interior regularity.

THEOREM 4.4.5. *Let $m \ge 0$ and $g \in C^m(\mathbb{R},\mathbb{R})$. If $u \in L^\infty_{\mathrm{loc}}(\Omega)$ satisfies $-\Delta u = g(u)$ in $\mathcal{D}'(\Omega)$, then $u \in W^{m+2,p}_{\mathrm{loc}}(\Omega) \cap C^{m+1,\alpha}_{\mathrm{loc}}(\Omega)$ for all $1 < p < \infty$ and all $0 \le \alpha < 1$. In particular, if $g \in C^\infty(\mathbb{R},\mathbb{R})$, then $u \in C^\infty(\Omega)$.*

For the proof of Theorem 4.4.5, we will use the following estimate.

PROPOSITION 4.4.6. *Let $m \ge 1$ and $g \in C^m(\mathbb{R},\mathbb{R})$ such that $g(0) = 0$. Let $1 \le p < \infty$. It follows that $g(u) \in W^{m,p}(\mathbb{R}^N)$ for all $u \in W^{m,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Moreover, given any $M > 0$, there exists a constant $C(M)$ such that*

$$\|g(u)\|_{W^{m,p}} \le C(M)\|u\|_{W^{m,p}}, \tag{4.4.13}$$

*for all $u \in W^{m,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $\|u\|_{L^\infty} \le M$.*

PROOF. We fix $M > 0$, so that we may without loss of generality modify $g(t)$ for $|t| > M$. Considering for example a function $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta(t) = 1$ for $|t| \le M$, we may replace $g(t)$ by $\eta(t)g(t)$, so that we may assume

$$\sup_{t\in\mathbb{R}} \sup_{0\le j\le m} |g^{(j)}(t)| < \infty. \tag{4.4.14}$$

Next, we observe that if $0 \le |\beta| \le \ell$, then it follows from Gagliardo-Nirenberg's inequality (5.4.2) (in fact, the simpler interpolation inequality (5.4.25) is sufficient) that there exists a constant $C$ such that

$$|u|_{|\beta|,\frac{\ell p}{|\beta|}} \le C|u|_{\ell,p}^{\frac{|\beta|}{\ell}} \|u\|_{L^\infty}^{\frac{\ell-|\beta|}{\ell}}, \tag{4.4.15}$$

for all $u \in C_c^\ell(\mathbb{R}^N)$. We now proceed in three steps.

STEP 1. $L^p$ estimates. It follows from (4.4.14) that $|g(t)| \le C|t|$, from which we deduce that

$$\|g(u)\|_{L^p} \le C\|u\|_{L^p}, \tag{4.4.16}$$

for all $u \in L^p(\mathbb{R}^N)$. Next, since $|g(t) - g(s)| \le C|t-s|$ by (4.4.14), we see that

$$\|g(u) - g(v)\|_{L^p} \le C\|u - v\|_{L^p}, \tag{4.4.17}$$

for all $u \in L^p(\mathbb{R}^N)$.

STEP 2. The case $u \in C_c^\infty(\mathbb{R}^N)$. Let $u \in C_c^\infty(\mathbb{R}^N)$ with $\|u\|_{L^\infty} \le M$. Let $1 \le \ell \le m$ and consider a multi-index $\alpha$ with $|\alpha| = \ell$. It is not difficult to show that $D^\alpha g(u)$ is a sum of terms of the form

$$g^{(k)}(u) \prod_{j=1}^{k} D^{\beta_j} u, \tag{4.4.18}$$

where $k \in \{1, \dots, \ell\}$ and the $\beta_j$'s are multi-indices such that $\alpha = \beta_1 + \dots + \beta_k$ and $|\beta_j| \ge 1$. Let $p_j = \ell p / |\beta_j|$, so that

$$\sum_{j=1}^{k} \frac{1}{p_j} = \frac{1}{p}. \tag{4.4.19}$$

It follows from (4.4.19), Hölder's inequality and (4.4.15) that

$$\| \prod_{j=1}^{k} D^{\beta_j} u \|_{L^p} \le \prod_{j=1}^{k} \|D^{\beta_j} u\|_{L^{p_j}} \le C(M) \|u\|_{W^{\ell,p}}. \tag{4.4.20}$$

We deduce from (4.4.16), (4.4.18) and (4.4.20) that (4.4.13) holds for all $u \in C_c^\infty(\mathbb{R}^N)$.

STEP 3. Conclusion. Let $u \in W^{m,p}(\mathbb{R}^N)$ with $\|u\|_{L^\infty} \le M$ and let $(u_n)_{n\ge0} \subset C_c^\infty(\mathbb{R}^N)$ with $u_n \to u$ in $W^{m,p}(\mathbb{R}^N)$. We note that one can construct the sequence $(u_n)_{n\ge0}$ by truncation and regularization, so that we may also assume that $\|u_n\|_{L^\infty} \le M$. Therefore, it follows from Step 2 that

$$\|g(u_n)\|_{W^{m,p}} \le C(M) \|u_n\|_{W^{m,p}}. \tag{4.4.21}$$

Since $g(u_n) \to g(u)$ in $L^p(\mathbb{R}^N)$ by (4.4.17), it follows from (4.4.21) that $g(u) \in W^{m,p}(\mathbb{R}^N)$ and that $g(u_n) \rightharpoonup g(u)$ in $W^{m,p}(\mathbb{R}^N)$. (See Lemma 5.5.3.) Letting $n \to \infty$ in (4.4.21), we obtain (4.4.13). □

COROLLARY 4.4.7. *Let $m \ge 0$ and $g \in C^m(\mathbb{R}, \mathbb{R})$. Let $\Omega$ be an open domain of $\mathbb{R}^N$. If $u \in W^{m,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ for some $1 \le p < \infty$, then $g(u) \in W^{m,p}_{\mathrm{loc}}(\Omega)$.*

PROOF. The result is immediate for $m = 0$, so we assume $m \ge 1$. Let $\omega \subset\subset \Omega$ and let $\varphi \in C_c^\infty(\Omega)$ satisfy $\varphi(x) = 1$ for all $x \in \omega$. Set

$$v(x) = \begin{cases} \varphi(x)u(x) & \text{if } \; x \in \Omega, \\ 0 \quad \text{if } \; x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

so that $v \in W^{m,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Letting $h(t) = g(t) - g(0)$, we deduce from Proposition 4.4.6 that $h(v) \in W^{m,p}(\mathbb{R}^N)$. Since $h(v) = h(u) = g(u) - g(0)$ in $\omega$, we deduce that $g(u) \in W^{m,p}(\omega)$. Hence the result, since $\omega \subset\subset \Omega$ is arbitrary. □

PROOF OF THEOREM 4.4.5. Fix $1 < p < \infty$. Let $0 \le j \le m$ and assume that $u \in W^{j,p}_{\mathrm{loc}}(\Omega)$. (This is certainly true for $j = 0$.) It follows from Corollary 4.4.7 that $g(u) \in W^{j,p}_{\mathrm{loc}}(\Omega)$, so that $u \in W^{j+2,p}_{\mathrm{loc}}(\Omega)$ by Proposition 4.1.2 (i). By induction on $j$, we deduce that $u \in W^{m+2,p}_{\mathrm{loc}}(\Omega)$ for all $1 < p < \infty$. Let now $\omega \subset\subset \Omega$ and consider a function $\varphi \in C_c^\infty(\Omega)$ such that $\varphi = 1$ on $\omega$. It follows that $v = \varphi u \in W_0^{m+2,p}(\Omega)$ for all $1 < p < \infty$. Applying Theorem 5.4.16, we deduce that $v \in C^{m+1,\frac{p-N}{p}}(\overline{\Omega})$ for all $N < p < \infty$. Since $v = u$ in $\omega$ and $\omega \subset\subset \Omega$ is arbitrary, the result follows. □

We end this section with two results concerning the case $\Omega = \mathbb{R}^N$. One is the an higher-order global regularity result, and the other an exponential decay property.

THEOREM 4.4.8. *Let $m \ge 1$ and $g \in C^m(\mathbb{R},\mathbb{R})$ satisfy $g(0)=0$. Let $u \in L^r(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ for some $1<r<\infty$. If $u$ satisfies $-\Delta u = g(u)$ in $\mathcal{D}'(\mathbb{R}^N)$, then $u \in W^{m+2,p}(\mathbb{R}^N)$ for all $r \le p < \infty$. In particular, $u \in C_0^{m+1}(\mathbb{R}^N)\cap C^{m+1,\alpha}(\mathbb{R}^N)$ for all $0<\alpha<1$.*

PROOF. Fix $r \le p < \infty$. Let $0 \le j \le m$ and assume that $u \in W^{j,p}(\mathbb{R}^N)$. (This is certainly true for $j=0$.) It follows from Proposition 4.4.6 (if $j \ge 1$; or a direct calculation based on the property $g \in C^1$ and $g(0)=0$ if $j=0$) that $g(u) \in W^{j,p}(\mathbb{R}^N)$, so that $u \in W^{j+2,p}(\mathbb{R}^N)$ by Proposition 4.1.1. By induction on $j$, we deduce that $u \in W^{m+2,p}(\mathbb{R}^N)$ for all $r \le p < \infty$. The last property follows from Theorem 5.4.16. □

PROPOSITION 4.4.9. *Let $g \in C^1(\mathbb{R},\mathbb{R})$. Assume $g(0)=0$, $g'(0)<0$ and let $u \in L^r(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ for some $1<r<\infty$. If $u$ satisfies $-\Delta u = g(u)$ in $\mathcal{D}'(\mathbb{R}^N)$, then there exists $\varepsilon>0$ such that $\sup\limits_{x\in\mathbb{R}^N} e^{\varepsilon|x|}(|u(x)|+|\nabla u(x)|)<\infty$.*

PROOF. We set $\delta = \sqrt{-g'(0)} > 0$ and $h(t) = g(t)+\delta^2 t$, so that

$$\frac{h(t)}{t} \underset{t\to 0}{\longrightarrow} 0, \tag{4.4.22}$$

and

$$-\Delta u + \delta^2 u = h(u). \tag{4.4.23}$$

Given $\varepsilon>0$, we set $\varphi_\varepsilon(x) = e^{\frac{\delta|x|}{1+\varepsilon|x|}}$. One verifies easily that $\varphi_\varepsilon \in W^{1,\infty}(\mathbb{R}^N)$ and that

$$|\nabla\varphi_\varepsilon| \le \delta\varphi_\varepsilon. \tag{4.4.24}$$

Next, we note that $u \in W^{3,q}(\mathbb{R}^N)$ for all $p \le q < \infty$ by Theorem 4.4.8. In particular, $u$ and $\nabla u$ are globally Lipschitz continuous and $|u(x)|+|\nabla u(x)| \to 0$ as $|x|\to\infty$. Moreover, the equation (4.4.23) makes sense in $L^p(\mathbb{R}^N)$. Since $\varphi_\varepsilon u \in W^{1,1}(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$, and in particular $\varphi_\varepsilon u \in W^{1,p'}(\mathbb{R}^N)$, we obtain by multiplying the equation by $\varphi_\varepsilon u$ and integrating by parts (see formula (5.1.5))

$$\begin{aligned}\int_{\mathbb{R}^N}\varphi_\varepsilon|\nabla u|^2+\delta^2\int_{\mathbb{R}^N}\varphi_\varepsilon u^2 &= \int_{\mathbb{R}^N}\varphi_\varepsilon u h(u) - \int_{\mathbb{R}^N} u\nabla u\cdot\nabla\varphi_\varepsilon \\ &\le \int_{\mathbb{R}^N}\varphi_\varepsilon u h(u) + \frac12\int_{\mathbb{R}^N}\varphi_\varepsilon|\nabla u|^2 + \frac{\delta^2}{2}\int_{\mathbb{R}^N}\varphi_\varepsilon u^2,\end{aligned}$$

where we used (4.4.24) and Cauchy-Schwarz in the last inequality. Thus we see that

$$\int_{\mathbb{R}^N}\varphi_\varepsilon|\nabla u|^2 + \delta^2\int_{\mathbb{R}^N}\varphi_\varepsilon u^2 \le 2\int_{\mathbb{R}^N}\varphi_\varepsilon u h(u). \tag{4.4.25}$$

Next, we deduce from (4.4.22) that there exists $\eta>0$ such that $4th(t) \le \delta^2 t^2$ for all $|t|\le\eta$. Also, since $u(x)\to 0$ as $|x|\to\infty$, there exists $R>0$ such that $|u(x)|\le\eta$ for $|x|\ge R$. Thus we see that $2uh(u) \le (\delta^2/2)u^2$ for $|x|\ge R$, so that

$$2\int_{\mathbb{R}^N}\varphi_\varepsilon u h(u) \le 2\int_{\{|x|<R\}}\varphi_\varepsilon u h(u) + \frac{\delta^2}{2}\int_{\{|x|>R\}}\varphi_\varepsilon u^2.$$

Applying now (4.4.25), it follows that

$$\begin{aligned}\int_{\mathbb{R}^N}\varphi_\varepsilon|\nabla u|^2 + \delta^2\int_{\mathbb{R}^N}\varphi_\varepsilon u^2 &\le 4\int_{\{|x|<R\}}\varphi_\varepsilon u h(u) \\ &\le 4\int_{\{|x|<R\}} e^{\delta|x|}|u|\,|h(u)| < \infty.\end{aligned}$$

Finally, letting $\varepsilon \downarrow 0$, we deduce that

$$\int_{\mathbb{R}^N} e^{\delta|x|}|\nabla u|^2 + \delta^2 \int_{\mathbb{R}^N} e^{\delta|x|}u^2 < \infty.$$

Since both $u$ and $\nabla u$ are globally Lipschitz continuous, the exponential decay easily follows. □

### 4.5. Symmetry of positive solutions

In this section, we show that if $u \in H_0^1(\Omega)$ is a solution of the equation $-\Delta u = g(u)$ in $H^{-1}(\Omega)$, then $u$ inherits some of the symmetry properties of $\Omega$, under certain assumptions on $g$ and $u$. The main result of this section is the following theorem, due to Gidas, Ni and Nirenberg [22].

THEOREM 4.5.1. *Let $\Omega$ be the unit ball of $\mathbb{R}^N$. Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfy $-\Delta u = g(u)$ in $H^{-1}(\Omega)$. If $u > 0$ in $\Omega$, then $u$ is radially symmetric and decreasing in $r$.*

Since $\Omega$ is symmetric about every hyperplane of $\mathbb{R}^N$ containing 0, Theorem 4.5.1 is a consequence of the following more general result (and Remark 4.5.3 (ii)).

THEOREM 4.5.2. *Let $\Omega$ be an open, bounded, connected domain of $\mathbb{R}^N$. Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfy $-\Delta u = g(u)$ in $H^{-1}(\Omega)$. Suppose further that*

$$\Omega \text{ is convex in the } x_1 \text{ direction,} \tag{4.5.1}$$

*and*

$$\Omega \text{ is symmetric about the hyperplane } x_1 = 0. \tag{4.5.2}$$

*If $u > 0$ in $\Omega$, then $u$ is symmetric with respect to $x_1$ and decreasing in $x_1 > 0$.*

REMARK 4.5.3. Here are some comments on Theorem 4.5.2.

(i) The assumption (4.5.1) means that, given any $y \in \mathbb{R}^{N-1}$, the set $\{x_1 \in \mathbb{R};\ (x_1, y) \in \Omega\}$ is convex (i.e. is an interval). The assumption (4.5.2) means that, given any $x_1 \in \mathbb{R}$ and $y \in \mathbb{R}^{N-1}$, if $(x_1, y) \in \Omega$, then $(-x_1, y) \in \Omega$.

(ii) Clearly, one can replace the direction $x_1$ by any arbitrary direction in $S^{N-1}$.

(iii) If $g(0) \ge 0$ and if $u \ge 0$ in $\Omega$, then it follows from the strong maximum principle that if $u \not\equiv 0$, then $u > 0$ in $\Omega$. Thus if $g(0) \ge 0$, the assumptions of Theorem 4.5.2 can be weakened in the sense that we need only assume $u \not\equiv 0$ and $u \ge 0$ in $\Omega$.

(iv) The conclusion of Theorem 4.5.2 can be false if $\Omega$ is not convex in the $x_1$ direction or if $u$ is not positive in $\Omega$. Here are two simple one-dimensional examples. Let $\Omega = (-1, 0) \cup (0, 1)$. We see that $\Omega$ is symmetric about $x = 0$ but $\Omega$ is not convex. Let $u$ be defined by $u(x) = \sin \pi x$ if $0 < x < 1$ and $u(x) = 2\sin(-\pi x)$ if $-1 < x < 0$. It follows that $u > 0$ in $\Omega$ and that $u \in H_0^1(\Omega)$ satisfies the equation $-\Delta u = \pi^2 u$. However, $u$ is not even. Let now $\Omega = (-1, 1)$, so that $\Omega$ satisfies both (4.5.1) and (4.5.2). and let $u(x) = \sin \pi x$. It follows that $u \in H_0^1(\Omega)$ and that $-\Delta u = \pi^2 u$. However, $u$ is neither positive in $\Omega$ nor even.

(v) We note that no regularity assumption is made on the set $\Omega$.

We follow the proof of Berestycki and Nirenberg [9], based on the "moving planes" technique of Alexandroff. We begin with the following lemma, which gives a lower estimate of $\lambda_1(-\Delta; \Omega)$ in terms of $|\Omega|$.

LEMMA 4.5.4. *There exists a constant $\alpha(N) > 0$ such that*

$$\lambda_1(-\Delta; \Omega) \ge \alpha(N)|\Omega|^{-\frac{2}{N}}, \tag{4.5.3}$$

*for every open domain* $\Omega \subset \mathbb{R}^N$ *with finite measure, where* $\lambda_1(-\Delta;\Omega)$ *is defined by* (2.1.5)*.*

PROOF. It follows from Poincaré's inequality (5.4.73) that

$$\|u\|_{L^2}^2 \le C(N)|\Omega|^{\frac{2}{N}}\|\nabla u\|_{L^2}^2,$$

which implies (4.5.3) with $\alpha(N) = 1/C(N)$. □

LEMMA 4.5.5. *Let* $\Omega \subset \mathbb{R}^N$ *be an open domain and let* $g : \mathbb{R} \to \mathbb{R}$ *be globally Lipschitz continuous. If* $u, v \in L^1_{\mathrm{loc}}(\Omega)$*, then there exists* $a \in L^\infty(\Omega)$ *such that* $g(u) - g(v) = a(u - v)$ *a.e. in* $\Omega$ *and* $\|a\|_{L^\infty} \le L$ *where* $L$ *is the Lipschitz constant of* $g$*.*

PROOF. Let $a$ be defined a.e. by

$$a(x) = \begin{cases} \frac{g(u(x))-g(v(x))}{u(x)-v(x)} & \text{if} \quad u(x) \ne v(x), \\ 0 \quad \text{if} \quad u(x) = v(x), \end{cases}$$

and, given any $\varepsilon > 0$, let

$$a_\varepsilon = \frac{(u-v)(g(u)-g(v))}{\varepsilon + (u-v)^2}.$$

It is clear that $a_\varepsilon$ is measurable and that $|a_\varepsilon| \le L$ where $L$ is the Lipschitz constant of $g$. Moreover, $a_\varepsilon \to a$ a.e. as $\varepsilon \downarrow 0$, so that $a \in L^\infty(\Omega)$. Finally, $(u-v)a_\varepsilon(u-v) \to g(u) - g(v)$ a.e. as $\varepsilon \downarrow 0$, so that $g(u) - g(v) = a(u - v)$. □

PROOF OF THEOREM 4.5.2. We note that in dimension $N = 1$, the result follows from Remark 1.2.1. Thus we now assume $N \ge 2$. We first observe that, since $u \in L^\infty(\Omega)$, we may change without loss of generality $g(t)$ for all sufficiently large values of $|t|$. In particular, we may assume that $g$ is globally Lipschitz continuous and we denote by $L$ the Lipschitz constant of $g$. Next, we note that $u + g(u) \in L^\infty(\Omega)$, so that $u \in C(\Omega)$ by Corollary 4.2.7. We now introduce some notation. We denote by $\mathcal{P}$ the orthogonal projection on $\mathbb{R}^{N-1}$, i.e. if $x = (x_1, y) \in \mathbb{R}^N$, then

$$\mathcal{P}x = y.$$

Let

$$\mathcal{U} = \mathcal{P}\Omega = \{y \in \mathbb{R}^{N-1};\ \exists x_1 \in \mathbb{R}, (x_1, y) \in \Omega\}.$$

Given $y \in \mathcal{U}$, let

$$\rho(y) = \sup\{x_1 \in \mathbb{R};\ (x_1, y) \in \Omega\}.$$

It follows from the assumptions (4.5.1)-(4.5.2) that

$$\Omega = \bigcup_{y\in\mathcal{U}} (-\rho(y), \rho(y)) \times \{y\}. \tag{4.5.4}$$

In particular, we see that $y \in \mathcal{U}$ if and only if $(0, y) \in \Omega$, so that $\mathcal{U}$ is an open, bounded, connected subset of $\mathbb{R}^{N-1}$. Set

$$R = \sup\{\rho(y);\ y \in \mathcal{U}\}. \tag{4.5.5}$$

Given $0 \le \lambda \le R$, we define the open set

$$\Omega_\lambda = \{(x_1, y) \in \Omega;\ x_1 > \lambda\} = \bigcup_{y\in\mathcal{U}_\lambda} (\lambda, \rho(y)) \times \{y\}, \tag{4.5.6}$$

where

$$\mathcal{U}_\lambda = \mathcal{P}\Omega_\lambda = \{y \in \mathcal{U};\ \rho(y) > \lambda\}.$$

We see that $\Omega_\lambda \ne \emptyset$ for $0 \le \lambda < R$, that $\Omega_\lambda$ is decreasing in $\lambda \in [0, R]$, that $\Omega_R = \emptyset$ and that

$$|\Omega_\lambda| \underset{\lambda\uparrow R}{\longrightarrow} 0. \tag{4.5.7}$$

Moreover, it follows from (4.5.4)-(4.5.6) that, given any $(x_1, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$,

$$(x_1, y) \in \Omega_\lambda \Rightarrow (2\lambda - x_1, y) \in \Omega.$$

Given $0 \le \lambda < R$, we define the function $u_\lambda$ on $\Omega_\lambda$ by

$$u_\lambda(x_1, y) = u(2\lambda - x_1, y) - u(x_1, y), \quad \text{for all} \quad (x_1, y) \in \Omega_\lambda. \tag{4.5.8}$$

We then see that

$$u_\lambda \in H^1(\Omega_\lambda) \cap C(\Omega_\lambda),$$

and that $-\Delta u_\lambda = f_\lambda$ in $H^{-1}(\Omega_\lambda)$, where $f_\lambda(x_1, y) = g(u(2\lambda - x_1, y) - g(u(x_1, y))$ for all $(x_1, y) \in \Omega_\lambda$. Applying Lemma 4.5.5, we deduce that there exists a function $a_\lambda \in L^\infty(\Omega_\lambda)$ such that

$$\|a_\lambda\|_{L^\infty} \le L, \tag{4.5.9}$$

and $f_\lambda = a_\lambda u_\lambda$. Therefore,

$$-\Delta u_\lambda = a_\lambda u_\lambda, \tag{4.5.10}$$

in $H^{-1}(\Omega_\lambda)$. We now proceed in nine steps.

STEP 1. If $\omega$ is an open subset of $\Omega$, then $\mathcal{P}\omega$ is an open subset of $\mathcal{U}$. Indeed, let $y^0 \in \mathcal{P}\omega$ and fix $x_1^0 \in \mathbb{R}$ such that $(x_1^0, y^0) \in \omega$. $\omega$ being open, there exists $\varepsilon > 0$ such that if $|(x_1 - x_1^0, y - y^0)| < \varepsilon$ then $(x_1, y) \in \omega$. In particular, if $|y - y^0| < \varepsilon$ (where the norm is in $\mathbb{R}^{N-1}$), then $(x_1^0, y) \in \omega$, so that $y \in \mathcal{P}\omega$. Thus $\mathcal{P}\omega$ is an open subset of $\mathcal{U}$.

STEP 2. If $0 < \lambda < R$ and if $\omega \subset \Omega_\lambda$ is a connected component of $\Omega_\lambda$, then

$$\omega = \bigcup_{y \in \mathcal{O}} (\lambda, \rho(y)) \times \{y\}, \tag{4.5.11}$$

where $\mathcal{O} = \mathcal{P}\omega$. Indeed, let $(x_1^0, y^0) \in \omega$. It follows from (4.5.6) that $\lambda < x_1^0 < \rho(y^0)$ and that $(\lambda, \rho(y)) \times \{y^0\} \subset \Omega_\lambda$. Since $(\lambda, \rho(y)) \times \{y^0\}$ is a closed, connected subset of $\Omega_\lambda$, $\omega$ is a connected component of $\Omega_\lambda$, and $(\lambda, \rho(y)) \times \{y^0\} \cap \omega \ne \emptyset$, we see that $(\lambda, \rho(y)) \times \{y^0\} \subset \omega$. In particular, $\omega$ is given by (4.5.11).

STEP 3. For almost all $y \in \mathcal{U}$, $u(x_1, y) \to 0$ as $x_1 \uparrow \rho(y)$. Indeed, let $(u_n)_{n \ge 0} \subset C_c^\infty(\Omega)$ satisfy $u_n \to u$ in $H_0^1(\Omega)$. In particular, $u_n(\cdot, y) - u(\cdot, y) \in H^1(-\rho(y), \rho(y))$ for a.a. $y \in \mathcal{U}$ and

$$\|\partial_1 u_n - \partial_1 u\|_{L^2(\Omega)}^2 = \int_{\mathcal{U}} \|u_n(\cdot, y) - u(\cdot, y)\|_{H^1(-\rho(y), \rho(y))}^2 \, dy \underset{n \to \infty}{\longrightarrow} 0.$$

Therefore, up to a subsequence, $u_n(\cdot, y) \to u(\cdot, y)$ in $H^1(-\rho(y), \rho(y))$ for a.a. $y \in \mathcal{U}$. Since clearly $u_n(\cdot, y) \in H_0^1(-\rho(y), \rho(y))$ for all $y \in \mathcal{U}$, we deduce that $u(\cdot, y) \in H_0^1(-\rho(y), \rho(y))$ for a.a. $y \in \mathcal{U}$, and the result follows.

STEP 4. If $0 < \lambda < R$ and $\omega \subset \Omega_\lambda$ is a connected component of $\Omega_\lambda$, then $u_\lambda \not\equiv 0$ in $\omega$. Indeed, let $\mathcal{O} = \mathcal{P}(\omega)$ so that $\mathcal{O}$ is an open subset of $\mathcal{U}$ by Step 1. It follows from Step 3 that there exists $y \in \mathcal{O}$ such that $u(x_1, y) \to 0$ as $x_1 \uparrow \rho(y)$. We note that $-\rho(y) < 2\lambda - \rho(y) < \rho(y)$, so that $(2\lambda - \rho(y), y) \in \Omega$. In particular, $u(2\lambda - \rho(y), y) > 0$ and $u(2\lambda - x_1, y) \to u(2\lambda - \rho(y), y)$ as $x_1 \uparrow \rho(y)$. Thus we see that $u_\lambda(x_1, y) \to u(2\lambda - \rho(y), y) > 0$ as $x_1 \uparrow \rho(y)$. Since $(x_1, y) \in \omega$ for $\rho(y) - x_1$ sufficiently small by (4.5.11), the result follows.

STEP 5. If $0 < \lambda < R$, then $u_\lambda^- \in H_0^1(\Omega_\lambda)$. Indeed, let $(u_n)_{n \ge 0} \subset C_c^\infty(\Omega)$ satisfy $u_n \ge 0$ and $u_n \to u$ in $H_0^1(\Omega)$ (see the beginning of Section 3.1). It follows easily that $(u_n)_\lambda \to u_\lambda$ in $H^1(\Omega_\lambda)$, so that $(u_n)_\lambda^- \to u_\lambda^-$ in $H^1(\Omega_\lambda)$. It thus suffices to show that $(u_n)_\lambda^- \in H_0^1(\Omega_\lambda)$ for all $n \ge 0$. Since $(u_n)_\lambda^- \in C(\overline{\Omega_\lambda})$, we need only show that $(u_n)_\lambda^-$ vanishes on $\partial\Omega_\lambda$ (see Remark 5.1.10 (ii)). It is not difficult to show that

$$\begin{aligned} \partial\Omega_\lambda &= \Big(\partial\Omega \cap \{(x_1, y) \in \mathbb{R}^N;\, x_1 > \lambda\}\Big) \cup \Big(\bigcup_{y \in \overline{\mathcal{U}_\lambda}} \{(\lambda, y)\}\Big) \\ &=: A \cup B. \end{aligned} \tag{4.5.12}$$

If $x \in A$, then $u_n(x) = 0$ since $x \in \partial\Omega$, so that $(u_n)_\lambda(x) \geq 0$. Thus $(u_n)_\lambda^-(x) = 0$. If $x \in B$, then $(u_n)_\lambda(x) = u(x) - u(x) = 0$, so that $(u_n)_\lambda^-(x) = 0$. Thus we see that $(u_n)_\lambda^-(x) = 0$ for all $x \in \partial\Omega_\lambda$, which proves the desired result.

STEP 6. If $0 < \lambda < R$ and $u_\lambda \geq 0$ in $\Omega_\lambda$, then $u_\lambda > 0$ in $\Omega_\lambda$. Indeed, it follows from (4.5.9)-(4.5.10) that $-\Delta u_\lambda + L u_\lambda \geq 0$ in $H^{-1}(\Omega_\lambda)$. Using Steps 4 and 5, we may apply the strong maximum principle in every connected component of $\Omega_\lambda$ and the result follows.

STEP 7. There exists $0 < \delta < R$ such that if $R - \delta < \lambda < R$, then $u_\lambda > 0$ in $\Omega_\lambda$. Indeed, we observe that

$$\lambda_1(-\Delta - a_\lambda; \Omega_\lambda) \geq \lambda_1(-\Delta; \Omega_\lambda) - \|a_\lambda\|_{L^\infty} \geq \alpha(N)|\Omega_\lambda|^{-\frac{2}{N}} - L,$$

by Lemma 4.5.4 and (4.5.9). Applying now (4.5.7), we see that there exists $\delta > 0$ such that $\lambda_1(-\Delta - a_\lambda) > 0$ for $R - \delta < \lambda < R$. Therefore, we deduce from Step 5 and the maximum principle that $u_\lambda \geq 0$ in $\Omega_\lambda$, and the conclusion follows from Step 6.

STEP 8. $u_\lambda > 0$ in $\Omega_\lambda$ for all $0 < \lambda < R$. To see this, we set

$$\mu = \inf\{0 < \sigma < R;\ u_\lambda > 0 \text{ in } \Omega_\lambda \text{ for all } \sigma < \lambda < R\},$$

so that $0 \leq \mu < R$ by Step 7. The conclusion follows if we show that $\mu = 0$. Suppose by contradiction that $\mu > 0$. Since $u \in C(\Omega)$, we see by letting $\lambda \downarrow \mu$ that $u_\mu \geq 0$ in $\Omega_\mu$, and it follows from Step 6 that $u_\mu > 0$ in $\Omega_\mu$. Let $K \subset \Omega_\mu$ be a closed set such that

$$\alpha(N)|\Omega_\mu \setminus K|^{-\frac{2}{N}} \geq 2L, \tag{4.5.13}$$

where $\alpha(N)$ is given by Lemma 4.5.4. Since $u_\mu \geq \eta > 0$ on $K$ by compactness, we see that there exists $0 < \delta < \mu$ such that

$$u_\nu > 0 \quad \text{on} \quad K, \tag{4.5.14}$$

for $\mu - \delta \leq \nu \leq \mu$. On the other hand, by choosing $\delta > 0$ possibly smaller, we deduce from (4.5.13) that

$$\alpha(N)|\Omega_\nu \setminus K|^{-\frac{2}{N}} > L, \tag{4.5.15}$$

for $\mu - \delta \leq \nu \leq \mu$. In particular,

$$\begin{aligned}\lambda_1(-\Delta + a_\nu; \Omega_\nu \setminus K) &\geq \lambda_1(-\Delta; \Omega_\nu \setminus K) - \|a_\nu\|_{L^\infty} \\ &\geq \alpha(N)|\Omega_\nu \setminus K|^{-\frac{2}{N}} - L > 0,\end{aligned} \tag{4.5.16}$$

by Lemma 4.5.4, (4.5.9) and (4.5.15). We claim that

$$u_\nu^- \in H_0^1(\Omega_\nu \setminus K). \tag{4.5.17}$$

To see this, we observe that by (4.5.14), $u_\nu^-$ vanishes in a neighborhood of $K$. Therefore, there exists a function $\theta \in C_c^\infty(\mathbb{R}^N \setminus K)$ such that $\theta u_\nu^- = u_\nu^-$. Consider a sequence $(\varphi_n)_{n\geq 0} \subset C_c^\infty(\Omega_\nu)$ such that $\varphi_n \to u_\nu^-$ in $H_0^1(\Omega_\nu)$. Since $\theta\varphi_n$ is supported in a compact subset of $\Omega_\nu \setminus K$, we see that $\theta\varphi_n \in H_0^1(\Omega_\nu \setminus K)$; and since $\theta\varphi_n \to \theta u_\nu^- = u_\nu^-$ in $H^1(\Omega_\nu \setminus K)$, the claim (4.5.17) follows. It now follows from (4.5.16), (4.5.17) and the maximum principle that $u_\nu \geq 0$ in $\Omega_\nu \setminus K$. Applying (4.5.14), we deduce that $u_\nu \geq 0$ in $\Omega_\nu$, so that $u_\nu > 0$ in $\Omega_\nu$ by Step 6. This contradicts the definition of $\mu$.

STEP 9. Conclusion. We deduce in particular from Step 8 (by letting $\lambda \downarrow 0$) that $u(x_1, y) \geq u(-x_1, y)$ for all $(x_1, y) \in \Omega$ with $x_1 \geq 0$. Changing $u(x_1, y)$ to $u(-x_1, y)$, we also have the reverse inequality, so that $u$ is symmetric with respect to $x_1$. Moreover, we deduce from Step 8 that if $(x_1, y) \in \Omega$ with $x_1 > 0$, then $u(2\lambda - x_1, y) > u(x_1, y)$ for all $0 < \lambda < x_1$. In particular, if $0 < x_1' < x_1$, then letting $\lambda = (x_1 + x_1')/2 < x_1$ we obtain $u(x_1, y) < u(x_1', y)$. This proves that $u$ is decreasing in $x_1 > 0$. □

REMARK 4.5.6. Part of the technicalities in the proof of Theorem 4.5.2 come from the fact that we do not assume that $u \in C_0(\Omega)$. Indeed, if $u \in C_0(\Omega)$, then Steps 3, 4, 5 and the end of Step 8 are trivial. However, since we did not make any smoothness assumption on $\Omega$, it is not clear how one could deduce the property $u \in C_0(\Omega)$ from standard regularity results.

CHAPTER 5

# Appendix: Sobolev spaces

Throughout this section, $\Omega$ is an open subset of $\mathbb{R}^N$. We study the basic properties of the Sobolev spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$, and particularly the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ (which correspond to $m = 1$ and $p = 2$). For a more detailed study, see for example Adams [1] .

## 5.1. Definitions and basic properties

We begin with the definition of "weak" derivatives. Let $u \in C^m(\Omega)$, $m \geq 1$. If $\alpha \in \mathbb{N}^N$ is a multi-index such that $|\alpha| \leq m$, it follows from Green's formula that

$$\int_\Omega D^\alpha u \varphi = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi, \tag{5.1.1}$$

for all $\varphi \in C_c^m(\Omega)$. We note that both integrals in (5.1.1) make sense since $D^\alpha u \varphi \in C_c(\Omega)$ and $u D^\alpha \varphi \in C_c(\Omega)$. As a matter of fact, the right-hand side makes sense as soon as $u \in L^1_{\mathrm{loc}}(\Omega)$ and the left-hand side makes sense as soon as $D^\alpha u \in L^1_{\mathrm{loc}}(\Omega)$. This motivates the following definition.

Definition 5.1.1. Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and let $\alpha \in \mathbb{N}^N$. We say that $D^\alpha u \in L^1_{\mathrm{loc}}(\Omega)$ if there exists $u_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u_\alpha \varphi = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi, \tag{5.1.2}$$

for all $\varphi \in C_c^{|\alpha|}(\Omega)$. Such a function $u_\alpha$ is then unique and we set $D^\alpha u = u_\alpha$. If $u_\alpha \in L^p_{\mathrm{loc}}(\Omega)$ (respectively, $u \in L^p(\Omega)$) for some $1 \leq p \leq \infty$, we say that $D^\alpha u \in L^p_{\mathrm{loc}}(\Omega)$ (respectively, $D^\alpha u \in L^p(\Omega)$).

The Sobolev spaces $W^{m,p}(\Omega)$ are defined as follows.

Definition 5.1.2. Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. We set

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega);\ D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leq m\}.$$

For $u \in W^{m,p}(\Omega)$, we set

$$\|u\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p},$$

which defines a norm on $W^{m,p}(\Omega)$. We set

$$H^m(\Omega) = W^{m,2}(\Omega),$$

and we equip $H^m(\Omega)$ with the scalar product

$$(u, v)_{H^m} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v \, dx,$$

which defines on $H^m(\Omega)$ the norm

$$\|u\|_{H^m} = \Big( \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^2}^2 \Big)^{\frac{1}{2}},$$

which is equivalent to the norm $\|\cdot\|_{W^{m,2}}$.

PROPOSITION 5.1.3. *$W^{m,p}(\Omega)$ is a Banach space and $H^m(\Omega)$ is a Hilbert space. If $p<\infty$, then $W^{m,p}(\Omega)$ is separable, and if $1<p<\infty$, then $W^{m,p}(\Omega)$ is reflexive.*

PROOF. Let $k=1+N+\cdots+N^m=(N^{m+1}-1)/(N-1)$ ($k=m+1$ if $N=1$), and consider the operator $T:W^{m,p}(\Omega)\to L^p(\Omega)^k$ defined by

$$Tu=(D^\alpha u)_{|\alpha|\le m}.$$

It is clear that $T$ is isometric and that $T$ is injective. Therefore, $W^{m,p}(\Omega)$ can be identified with the subspace $T(W^{m,p}(\Omega))$ of $L^p(\Omega)^k$.

We claim that $T(W^{m,p}(\Omega))$ is closed. Indeed, suppose $(u_n)_{n\ge 0}$ is such that $u_n\to u$ in $L^p(\Omega)$ and $D^\alpha u_n\to u_\alpha$ in $L^p(\Omega)$ for $1\le|\alpha|\le m$. Applying (5.1.1) to $u_n$ and letting $n\to\infty$, we deduce that $D^\alpha u\in L^p(\Omega)$ and that $D^\alpha u=u_\alpha$; and so, $u\in W^{m,p}(\Omega)$. Therefore, $T(W^{m,p}(\Omega))$ is a Banach space, and so is $W^{m,p}(\Omega)$. If $p<\infty$, then $L^p(\Omega)^k$ is separable. Thus so is $T(W^{m,p}(\Omega))$, hence $W^{m,p}(\Omega)$. Finally, if $1<p<\infty$, then $L^p(\Omega)^k$ is reflexive. Thus so is $T(W^{m,p}(\Omega))$, hence $W^{m,p}(\Omega)$. □

REMARK 5.1.4. Here are some simple consequences of Definition 5.1.2.

(i) It follows easily that if $u\in W^{m,p}(\Omega)$ and if $v\in C^m(\mathbb{R}^N)$ is such that $\sup\{\|D^\alpha v\|_{L^\infty};\,|\alpha|\le m\}<\infty$, then $uv\in W^{m,p}(\Omega)$ and Leibnitz formula holds.
(ii) If $|\Omega|<\infty$, then $L^p(\Omega)\hookrightarrow L^q(\Omega)$ provided $p\ge q$. It follows that $W^{m,p}(\Omega)\hookrightarrow W^{m,q}(\Omega)$.

REMARK 5.1.5. One can show that if $p<\infty$, then $W^{m,p}(\Omega)\cap C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$ (see Adams [1], Theorem 3.16 p. 52).

We now define the subspaces $W_0^{m,p}(\Omega)$ for $p<\infty$. Formally, $W_0^{m,p}(\Omega)$ is the subspace of functions of $W^{m,p}(\Omega)$ that vanish, as well of their derivatives up to order $m-1$, on $\partial\Omega$.

DEFINITION 5.1.6. Let $1\le p<\infty$ and let $m\in\mathbb{N}$. We denote by $W_0^{m,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$, and we set $H_0^m(\Omega)=W_0^{m,2}(\Omega)$.

REMARK 5.1.7. It follows from Proposition 5.1.3 that $W_0^{m,p}(\Omega)$ is a separable Banach space and that $W_0^{m,p}(\Omega)$ is reflexive if $p>1$. In addition, $H_0^m(\Omega)$ is a separable Hilbert space.

In general $W_0^{m,p}(\Omega)\neq W^{m,p}(\Omega)$, however both spaces coincide when $\partial\Omega$ is "small" (see Adams [1], Sections 3.20–3.33). In particular, we have the following result.

THEOREM 5.1.8. *If $1\le p<\infty$ and $m\in\mathbb{N}$, then $W_0^{m,p}(\mathbb{R}^N)=W^{m,p}(\mathbb{R}^N)$.*

The proof of Theorem 5.1.8 makes use of the following lemma.

LEMMA 5.1.9. *Let $\rho\in C_c^\infty(\mathbb{R}^N)$, $\rho\ge 0$, with $\operatorname{supp}\rho\subset\{x\in\mathbb{R}^N;\,|x|\le 1\}$ and $\|\rho\|_{L^1}=1$. For $n\in\mathbb{N}$, $n\ge 1$, set $\rho_n(x)=n^N\rho(nx)$. ($(\rho_n)_{n\ge 1}$ is called a smoothing sequence.) Then the following properties hold.*

(i) *For every $u\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $\rho_n\star u\in C^\infty(\mathbb{R}^N)$.*
(ii) *If $u\in L^p(\mathbb{R}^N)$ for some $p\in[1,\infty]$, then $\rho_n\star u\in L^p(\mathbb{R}^N)$ and $\|\rho_n\star u\|_{L^p}\le\|u\|_{L^p}$. If $p<\infty$ or if $p=\infty$ and $u\in C_{\mathrm{b,u}}(\mathbb{R}^N)$, then $\rho_n\star u\to u$ in $L^p(\mathbb{R}^N)$ as $n\to\infty$.*
(iii) *If $u\in W^{m,p}(\mathbb{R}^N)$ for some $p\in[1,\infty]$ and $m\in\mathbb{N}$, then $\rho_n\star u\in W^{m,p}(\mathbb{R}^N)$ and $D^\alpha(\rho_n\star u)=\rho_n\star D^\alpha u$ for $|\alpha|\le m$. In particular, if $p<\infty$ or if $p=\infty$*

*and* $D^\alpha u \in C_{\mathrm{b,u}}(\mathbb{R}^N)$ *for all* $|\alpha| \le m$*, then* $\rho_n \star u \to u$ *in* $W^{m,p}(\mathbb{R}^N)$ *as* $n \to \infty$*.*

PROOF. (i) Since

$$\rho_n \star u(x) = \int_{\mathbb{R}^N} \rho_n(x-y)u(y)\,dy,$$

it is clear that $\rho_n \star u \in C(\mathbb{R}^N)$. One deduces easily from the above formula that $D^\alpha(\rho_n \star u) = (D^\alpha \rho_n) \star u$, and the result follows.

(ii) The first part of property (ii) follows from Young's inequality, since

$$\|\rho_n\|_{L^1} = \int_{\mathbb{R}^N} \rho_n(x)\,dx = n^N \int_{\mathbb{R}^N} \rho(nx)\,dx = \int_{\mathbb{R}^N} \rho(y)\,dy = 1.$$

Consider now $u \in C_{\mathrm{b,u}}(\mathbb{R}^N)$ and set $u_n = \rho_n \star u$. We have

$$u_n(x) = \int_{\mathbb{R}^N} \rho_n(y)u(x-y)\,dy, \quad u(x) = \int_{\mathbb{R}^N} \rho_n(y)u(x)\,dy;$$

and so,

$$u_n(x) - u(x) = \int_{\mathbb{R}^N} \rho_n(y)(u(x-y) - u(x))\,dy.$$

Therefore,

$$|u_n(x) - u(x)| \le \int_{\mathbb{R}^N} \rho_n(y)|u(x-y) - u(x)|\,dy \le \sup_{|y|\le 1/n} |u(x-y) - u(x)|,$$

since $\operatorname{supp} \rho_n \subset \{y;\, |y| \le 1/n\}$. Since $u$ is uniformly continuous, we have

$$\sup_{x\in\mathbb{R}^N} \sup_{|y|\le 1/n} |u(x-y) - u(x)| \underset{n\to\infty}{\longrightarrow} 0;$$

and so, $u_n \to u$ uniformly. Consider next $u \in L^p(\mathbb{R}^N)$, with $p < \infty$, and let $\varepsilon > 0$. There exists $v \in C_{\mathrm{c}}(\mathbb{R}^N)$ such that $\|u - v\|_{L^p} \le \varepsilon/3$. Furthermore, it follows from what precedes that for $n$ large enough, we have $\|v - \rho_n \star v\|_{L^p} \le \varepsilon/3$. (Since $\rho_n \star v \to v$ uniformly and $\rho_n \star v$ is supported in a fixed compact subset of $\mathbb{R}^N$.) Finally, it follows from the inequality of (ii) that $\|\rho_n \star v - \rho_n \star u\|_{L^p} \le \|u - v\|_{L^p} \le \varepsilon/3$. Writing

$$u - \rho_n \star u = u - v + v - \rho_n \star v + \rho_n \star v - \rho_n \star u,$$

we deduce that $\|u - \rho_n \star u\|_{L^p} \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of property (ii).

(iii) For any $v : \mathbb{R}^N \to \mathbb{R}$, we set $\widetilde{v}(x) = v(-x)$. Given $u \in W^{m,p}(\mathbb{R}^N)$ and $\varphi \in C_{\mathrm{c}}^m(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} (\rho_n \star u) D^\alpha \varphi = \int_{\mathbb{R}^N} u(\widetilde{\rho_n} \star D^\alpha \varphi) = \int_{\mathbb{R}^N} u D^\alpha(\widetilde{\rho_n} \star \varphi).$$

By definition of $D^\alpha u$, we obtain

$$\int_{\mathbb{R}^N} (\rho_n \star u) D^\alpha \varphi = (-1)^{|\alpha|} \int_{\mathbb{R}^N} D^\alpha u(\widetilde{\rho_n} \star \varphi) = (-1)^{|\alpha|} \int_{\mathbb{R}^N} (\rho_n \star D^\alpha u)\varphi.$$

This means that $D^\alpha(\rho_n \star u) \in L^p(\mathbb{R}^N)$ and that $D^\alpha(\rho_n \star u) = \rho_n \star D^\alpha u$; and so, $\rho_n \star u \in W^{m,p}(\mathbb{R}^N)$. The convergence property follows from property (ii). □

PROOF OF THEOREM 5.1.8. Let $u \in W^{m,p}(\mathbb{R}^N)$ and $\varepsilon > 0$. It follows from properties (iii) and (i) of Lemma 5.1.9 that there exists $v \in W^{m,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ such that $\|u - v\|_{W^{m,p}} \le \varepsilon/2$. Fix now $\eta \in C_{\mathrm{c}}^\infty(\mathbb{R}^N)$ such that $0 \le \eta \le 1$, $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$. Set $\eta_n(x) = \eta(x/n)$ and let $v_n = \eta_n v$. It is clear that $v_n \in C_c^\infty(\mathbb{R}^N)$, and we claim that $\eta_n v \to v$ in $W^{m,p}(\mathbb{R}^N)$ as $n \to \infty$. Indeed, it follows from Leibnitz' formula (see Remark 5.1.4 (i)) that

$$D^\alpha(\eta_n v) = \sum_{\beta+\gamma=\alpha} D^\beta \eta_n D^\gamma u.$$

Since $\|D^\gamma \eta_n\|_{L^\infty} \le C_\gamma n^{-|\gamma|}$, it follows that all the terms with $|\beta| > 0$ converge to 0 as $n \to \infty$. The remaining term in the sum is $\eta_n D^\alpha v$ which, by dominated convergence, converges to $D^\alpha v$ in $L^p(\mathbb{R}^N)$. We deduce that $D^\alpha(\eta_n v) \to D^\alpha v$ in $L^p(\mathbb{R}^N)$, which proves the claim. Therefore, there exists $w \in C_c^\infty(\mathbb{R}^N)$ such that $\|v-w\|_{W^{m,p}} \le \varepsilon/2$. This implies that $\|u-w\|_{W^{m,p}} \le \varepsilon$, and the result follows. □

REMARK 5.1.10. We describe below some useful properties of the Sobolev space $W_0^{m,p}(\Omega)$.

(i) If $u \in W^{m,p}(\Omega)$ and if $\operatorname{supp} u$ is included in a compact subset of $\Omega$, then $u \in W_0^{m,p}(\Omega)$. This is easily shown by using the regularization and truncation argument described above.

(ii) If $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and if $u_{|\partial\Omega} = 0$, then $u \in W_0^{1,p}(\Omega)$. Indeed, if $u$ has a bounded support, let $F \in C^1(\mathbb{R})$ satisfy $|F(t)| \le |t|$, $F(t) = 0$ for $|t| \le 1$ and $F(t) = t$ for $|t| \ge 2$. Setting $u_n(x) = n^{-1}F(nu(x))$, it follows from Proposition 5.3.1 below that $u_n \in W^{1,p}(\Omega)$. In addition, one verifies easily (see (5.3.1) below) that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$. Since $\operatorname{supp} u_n \subset \{x \in \Omega;\ |u(x)| \ge n^{-1}\}$, $\operatorname{supp} u_n$ is a compact subset of $\Omega$, thus $u_n \in W_0^{1,p}(\Omega)$ by (i) above; and so $u \in W_0^{1,p}(\Omega)$. If $\operatorname{supp} u$ is unbounded, we approximate $u$ by $\xi_n u$ where $\xi_n \in C_c^\infty(\mathbb{R}^N)$ is such that $\xi_n(x) = 1$ for $|x| \le n$.

(iii) If $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ and if $\Omega$ is of class $C^1$, then $u_{|\partial\Omega} = 0$ (see Brezis [**11**], Théorème IX.17, p. 171). Note that this property is false if $\Omega$ is not smooth enough. For example, one can show that if $\Omega = \mathbb{R}^N \setminus \{0\}$ and $N \ge 2$, then $H_0^1(\Omega) = H^1(\Omega)$. In particular, if $u \in C_c^\infty(\mathbb{R}^N)$ and $u(0) \ne 0$, then $u \in H_0^1(\Omega)$ but $u \ne 0$ on $\partial\Omega$.

(iv) Let $u \in L^1_{\mathrm{loc}}(\Omega)$ and define $\widetilde{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ by

$$\widetilde{u}(x) = \begin{cases} u(x) \text{ if } x \in \Omega, \\ 0 \text{ if } x \notin \Omega. \end{cases}$$

If $u \in W_0^{1,p}(\Omega)$, then $\widetilde{u} \in W^{1,p}(\mathbb{R}^N)$. This is immediate by the definition of $W_0^{1,p}(\Omega)$. More generally, if $u \in W_0^{m,p}(\Omega)$, then $\widetilde{u} \in W^{m,p}(\mathbb{R}^N)$. Conversely, if $\widetilde{u} \in W^{1,p}(\mathbb{R}^N)$ and if $\Omega$ is of class $C^1$ (as in part (iii) above, the smoothness assumption on $\Omega$ is essential), then $u \in W_0^{1,p}(\Omega)$ (see Brezis [**11**], Proposition IX.18, p. 171).

PROPOSITION 5.1.11. *Let $1 \le p \le \infty$ and let $u \in W^{1,p}(\Omega)$. Let $\omega \subset \Omega$ be a connected, open set. If $\nabla u = 0$ a.e. on $\omega$, then there exists a constant $c$ such that $u = c$ a.e. on $\omega$.*

PROOF. Let $x \in \omega$ and let $\rho > 0$ be such that $B(x,\rho) \Subset \omega$. We claim that there exists $c$ such that $u = c$ a.e. on $B(x,\rho)$. The result follows by Connectedness. To prove the claim, we argue as follows. Let $0 < \varepsilon < \rho$ and let $\eta \in C_c^\infty(\mathbb{R}^N)$ satisfy $\eta \equiv 1$ on $B(x,\rho-\varepsilon)$, $\operatorname{supp}\eta \subset B(x,\rho)$, and $0 \le \eta \le 1$. Setting $v = \eta u$, we deduce that $v \in W_0^{1,1}(B(x,\rho))$ and that $\nabla v = 0$ a.e. on $B(x,\rho-\varepsilon)$. We now extend $v$ by 0 outside $B(x,\rho)$ and we call $\overline{v}$ the extension. Let $(\rho_n)_{n\ge0}$ be a smoothing sequence and fix $n > 1/(\rho-\varepsilon)$. We have $w_n = \rho_n \star \overline{v} \in C_c^\infty(\mathbb{R}^N)$. Furthermore, since $\nabla w_n = \rho_n \star \nabla\overline{v}$, and since $\operatorname{supp}\rho_n \subset B(0,1/n)$, it follows that $\nabla w_n = 0$ on $B(x,\rho-\varepsilon-1/n)$. In particular, there exists $c_n$ such that $w_n \equiv c_n$ on $B(x,\rho-\varepsilon-1/n)$. Since $w_n \to \overline{v}$ in $L^1(\mathbb{R}^N)$, we deduce in particular that for any $\mu < \rho - \varepsilon$, there exists $c(\mu)$ such that $\overline{v} \equiv c(\mu)$ on $B(x, \rho - \varepsilon - \mu)$. Therefore, $c(\mu)$ is independent of $\mu$ and we have $\overline{v} \equiv c$ on $B(x, \rho - \varepsilon)$. for some constant $c$. It follows that $c$ is independent of $\varepsilon$, and the claim follows by letting $\varepsilon \downarrow 0$. □

PROPOSITION 5.1.12. *Let $u \in W^{m,\infty}(\mathbb{R}^N)$ for some $m \geq 0$. If $D^\alpha u \in C_{b,u}(\mathbb{R}^N)$ for all $|\alpha| \leq m$, then $u \in C^m_{b,u}(\mathbb{R}^N)$. In other words, the distributional derivatives of $u$ are the classical derivatives.*

PROOF. Let $(\rho_n)_{n\geq 0}$ be a smoothing sequence and set $u_n = \rho_n \star u$. It follows from Lemma 5.1.9 that $u_n \in C^\infty(\mathbb{R}^N) \cap W^{m,\infty}(\mathbb{R}^N)$. Moreover, it is clear that $D^\alpha u_n = \rho_n \star (D^\alpha u)$ is uniformly continuous on $\mathbb{R}^N$ for all $|\alpha| \leq m$ and $n \geq 0$. Thus $(u_n)_{n\geq 0} \subset C^m_{b,u}(\mathbb{R}^N)$. Moreover, it follows from Lemma 5.1.9 that $D^\alpha u_n \to D^\alpha u$ in $L^\infty(\mathbb{R}^N)$, i.e. $u_n \to u$ in $W^{m,\infty}(\mathbb{R}^N)$. Since $C^m_{b,u}(\mathbb{R}^N)$ is a Banach space, it is a closed subset of $W^{m,\infty}(\mathbb{R}^N)$, and we deduce that $u \in C^m_{b,u}(\mathbb{R}^N)$. □

We next introduce the local Sobolev spaces.

DEFINITION 5.1.13. Given $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, we set $W^{m,p}_{\text{loc}}(\Omega) = \{u \in L^1_{\text{loc}}(\Omega);\, D^\alpha \in L^p_{\text{loc}}(\omega)$ for all $|\alpha| \leq m\}$.

PROPOSITION 5.1.14. *Let $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, let. If $u \in L^1_{\text{loc}}(\Omega)$, then the following properties are equivalent.*

(i) *$u \in W^{m,p}_{\text{loc}}(\Omega)$.*
(ii) *$u_{|\omega} \in W^{m,p}(\omega)$ for all $\omega \subset\subset \Omega$.*
(iii) *$\phi u \in W^{m,p}_0(\Omega)$ ($\varphi \in W^{m,\infty}(\Omega)$ if $p = \infty$) for all $\phi \in C^\infty_{\text{c}}(\Omega)$.*

PROOF. (i)⇒(ii). This is immediate.

(ii)⇒(iii). Suppose $u \in W^{m,p}_{\text{loc}}(\Omega)$ and let $\phi \in C^\infty_{\text{c}}(\Omega)$. If $\omega \subset\subset \Omega$ contains $\operatorname{supp}\phi$, then $\phi u$ has compact support in $\omega$. Since $u \in W^{m,p}(\omega)$, we know (see Remark 5.1.4 (i)) that $\phi u \in W^{m,p}(\omega)$. If $p < \infty$, then $\phi u \in W^{m,p}_0(\Omega)$ by Remark 5.1.10 (i).

(iii)⇒(i). Suppose $\phi u \in W^{m,p}(\Omega)$ for all $\phi \in C^\infty_{\text{c}}(\Omega)$. Given $|\alpha| \leq m$, we define $u_\alpha \in L^p_{\text{loc}}(\Omega)$ as follows. Let $\omega \subset\subset \Omega$ and let $\phi \in C^\infty_{\text{c}}(\Omega)$ satisfy $\phi(x) = 1$ for all $x \in \omega$. We set $(u_\alpha)_{|\omega} = D^\alpha(\phi u)_{|\omega}$ and we claim that $(u_\alpha)_{|\omega}$ is independent of the choice of $\phi$, so that $u_\alpha$ is well-defined. Indeed, if $\psi \in C^\infty_{\text{c}}(\Omega)$ is such that $\psi(x) = 1$ on $\omega$, then for all $\varphi \in C^\infty_{\text{c}}(\omega)$,

$$\int_\omega D^\alpha(\psi u - \phi u)\varphi = (-1)^{|\alpha|} \int_\omega (\psi - \phi) u D^\alpha \varphi = 0,$$

so that $D^\alpha \psi u = D^\alpha \phi u$ a.e. in $\omega$. It remains to show that $u_\alpha = D^\alpha u$. Indeed, let $|\alpha| \leq m$ and $\varphi \in C^{|\alpha|}_{\text{c}}(\Omega)$. Let $\phi \in C^\infty_{\text{c}}(\Omega)$ satisfy $\phi(x) = 1$ on $\operatorname{supp}\varphi$. We have

$$(-1)^{|\alpha|} \int_\Omega u_\alpha \varphi = \int_\Omega \phi u D^\alpha \varphi = \int_\Omega u D^\alpha \varphi,$$

and the result follows. □

We now introduce the Sobolev spaces of negative index.

DEFINITION 5.1.15. Given $m \in \mathbb{N}$ and $1 \leq p < \infty$, we define $W^{-m,p'}(\Omega) = (W^{m,p}_0(\Omega))^\star$. For $p = 2$, we set $H^{-m}(\Omega) = W^{-m,2}(\Omega) = (H^m_0(\Omega))^\star$.

REMARK 5.1.16. Here are some comments on Definition 5.1.15.

(i) It follows from Remark 5.1.7 that $W^{-m,p'}(\Omega)$ is a Banach space. If $p > 1$, then $W^{-m,p'}(\Omega)$ is reflexive and separable. $H^{-m}(\Omega)$ is a separable Hilbert space.

(ii) It follows from the dense embedding $C_c^\infty(\Omega) \hookrightarrow W_0^{m,p}(\Omega)$ that $W^{-m,p'}(\Omega)$ is a space of distributions on $\Omega$. In particular, we see that $(u,\varphi)_{W^{-m,p'},W_0^{m,p}} = (u,\varphi)_{\mathcal{D}',\mathcal{D}}$ for every $u \in W^{-m,p'}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$. Like any distribution, an element of $W^{-m,p'}(\Omega)$ can be localized. Indeed, if $u \in \mathcal{D}'(\Omega)$ and $\Omega'$ is an open subset of $\Omega$, then one defines $u_{|\Omega'}$ as follows. Given any $\varphi \in C_c^\infty(\Omega')$, let $\tilde{\varphi} \in C_c^\infty(\Omega)$ be equal to $\varphi$ on $\Omega'$ and to 0 on $\Omega \setminus \Omega'$. Then $\Psi(\varphi) = (u,\widetilde{\varphi})_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}$ defines a distribution $\Psi \in \mathcal{D}'(\Omega')$, and one sets $u_{|\Omega'} = \Psi$. Note that this is consistent with the usual restriction of functions. Since $\|\tilde{\varphi}\|_{W_0^{m,p}(\Omega')} \le \|\varphi\|_{W_0^{m,p}(\Omega)}$, we see that if $u \in W^{-m,p'}(\Omega)$, then $u_{|\Omega'} \in W^{-m,p'}(\Omega')$ and $\|u_{|\Omega'}\|_{W^{-m,p'}(\Omega')} \le \|u\|_{W^{-m,p'}(\Omega)}$.

Definition 5.1.17. Given $m \in \mathbb{N}$ and $1 \le p < \infty$, we define $W_{\text{loc}}^{-m,p'}(\Omega) = \{u \in \mathcal{D}'(\Omega);\ u_{|\omega} \in W^{-m,p'}(\omega) \text{ for all } \omega \subset\subset \Omega\}$. (See Remark 5.1.16 (ii) for the definition of $u_{|\omega}$.) For $p = 2$, we set $H_{\text{loc}}^{-m}(\Omega) = W_{\text{loc}}^{-m,2}(\Omega)$.

Proposition 5.1.18. *If $1 < p < \infty$, then $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W^{-1,p}(\Omega)$, with dense embeddings, where the embedding $e : L^p(\Omega) \to W^{-1,p}(\Omega)$ is defined by*

$$eu(\varphi) = \int_\Omega u(x)\varphi(x)\,dx, \tag{5.1.3}$$

*for all $\varphi \in W_0^{1,p'}(\Omega)$ and all $u \in L^p(\Omega)$.*

Proposition 5.1.18 is an immediate application of the following useful abstract result.

Proposition 5.1.19. *If $X$ and $Y$ are two Banach spaces such that $X \hookrightarrow Y$ with dense embedding, then, the following properties hold.*

(i) *$Y^\star \hookrightarrow X^\star$, where the embedding $e$ is defined by $(ef,x)_{X^\star,X} = (f,x)_{Y^\star,Y}$, for all $x \in X$ and $f \in Y^\star$.*
(ii) *If $X$ is reflexive, then the embedding $Y^\star \hookrightarrow X^\star$ is dense.*
(iii) *If the embedding $X \hookrightarrow Y$ is compact and $X$ is separable, then the embedding $Y^\star \hookrightarrow X^\star$ is compact. More precisely, if $(y'_n)_{n\ge0} \subset Y^\star$ and $\|y'_n\|_{Y^\star} \le M$, then there exist a subsequence $(n_k)_{k\ge0}$ and $y' \in Y^\star$ with $\|y'\|_{Y^\star} \le M$ such that $y'_{n_k} \to y'$ in $X^\star$ as $k \to \infty$.*

Proof. (i) Consider $y' \in Y^\star$ and $x \in X \hookrightarrow Y$. Let $ey'(x) = (y',x)_{Y^\star,Y}$. Since

$$|ey'(x)| \le \|y'\|_{Y^\star}\|x\|_Y \le C\|y'\|_{Y^\star}\|x\|_X,$$

we see that $e \in \mathcal{L}(Y^\star,X^\star)$. Suppose that $ey' = ez'$, for some $y',z' \in Y^\star$. It follows that $(y'-z',x)_{Y^\star,Y} = 0$, for every $x \in X$. By density, we deduce that $(y'-z',y)_{Y^\star,Y} = 0$, for every $y \in Y$; and so $y' = z'$. Thus $e$ is injective and (i) follows.

(ii) Assume to the contrary that $\overline{Y^\star} \neq X^\star$. Then there exists $x_0 \in X^{\star\star} = X$ such that $(y',x_0)_{X^\star,X} = 0$, for every $y' \in Y^\star$ (see e.g. Brezis [11], Corollary I.8). Let $E = \mathbb{R}x_0 \subset Y$, and let $f \in E^\star$ be defined by $f(\lambda x_0) = \lambda$, for $\lambda \in \mathbb{R}$. We have $\|f\|_{E^\star} = 1$, and by the Hahn-Banach theorem (see e.g. Brezis [11], Corollary I.2) there exists $y' \in Y^\star$ such that $\|y'\|_{Y^\star} = 1$ and $(y',x_0)_{Y^\star,Y} = 1$, which is a contradiction, since $(y',x_0)_{Y^\star,Y} = (y',x_0)_{X^\star,X} = 0$.

(iii) Let $B_{X^\star}$ (respectively, $B_X$, $B_{Y^\star}$, $B_Y$) be the unit ball of $X^\star$ (respectively, $X$, $Y^\star$, $Y$). Consider a sequence $(y'_n)_{n\ge0} \subset B_{Y^\star}$. Since $Y^\star$ is the dual of a separable Banach space, it follows (see e.g. Brezis [11], Corollary III.26) that there exist a subsequence, which we still denote by $(y'_n)_{n\ge0}$, and an element $y' \in B_{Y^\star}$ such that $y'_n \to y'$ in $Y^\star$ weak$^\star$. We show that $\|y'_n - y'\|_{X^\star} \to 0$, which proves the desired result. We note that

$$\begin{aligned} \|y_n' - y'\|_{X^\star} &= \sup_{x \in B_X} |(y_n' - y', x)_{X^\star,X}| \\ &= \sup_{x \in B_X} |(y_n' - y', x)_{Y^\star,Y}|, \end{aligned} \tag{5.1.4}$$

by (i). Let $\varepsilon > 0$. Since $B_X$ is a relatively compact subset of $Y$, we see that there exists a (finite) sequence $(x_j)_{1 \le j \le \ell} \subset B_X$ such that for every $x \in B_X$, there exists $1 \le j \le \ell$ such that $\|x - x_j\|_Y \le \varepsilon$. Given $x \in B_X$ and $1 \le j \le \ell$ as above, we deduce that

$$\begin{aligned} |(y_n' - y', x)_{Y^\star,Y}| &\le |(y_n' - y', x - x_j)_{Y^\star,Y}| + |(y_n' - y', x_j)_{Y^\star,Y}| \\ &\le \varepsilon \|y_n' - y'\|_{Y^\star} + |(y_n' - y', x_j)_{Y^\star,Y}| \\ &\le 2\varepsilon + |(y_n' - y', x_j)_{Y^\star,Y}|. \end{aligned}$$

Aplying now (5.1.4), we deduce that

$$\|y_n' - y'\|_{X^\star} \le 2\varepsilon + \sup_{1 \le j \le \ell} |(y_n' - y', x_j)_{Y^\star,Y}|.$$

Since $y_n' \to y'$ in $Y^\star$ weak$^\star$, we conclude that

$$\limsup_{n \to \infty} \|y_n' - y'\|_{X^\star} \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

REMARK 5.1.20. Proposition 5.1.18 calls for the following comments.

(i) Note that any Hilbert space can be identified, via the Riesz representation theorem, with its dual. By defining the embedding $e : L^2(\Omega) \to H^{-1}(\Omega)$ by (5.1.3), we implicitely identified $L^2(\Omega)$ with its dual. If we identify $H_0^1(\Omega)$ with its dual, so that $H^{-1}(\Omega) = H_0^1(\Omega)$, then Proposition 5.1.18 becomes absurd. This means that we cannot, at the same time, identify $L^2(\Omega)$ with its dual and $H_0^1(\Omega)$ with its dual, and use the canonical embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

(ii) Note that the density of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ can be viewed by a constructive argument (truncation and regularization). As well, any element $\varphi \in W_0^{-1,p}(\Omega)$ with compact support (in the sense that there exists a compact set $K$ of $\Omega$ such that $(\varphi, u)_{W^{-1,p}, W_0^{1,p'}} = 0$ for all $u \in W_0^{1,p'}(\Omega)$ supported in $\Omega \setminus K$) can be approximated by convolution by elements of $C_c^\infty(\Omega)$. However, it is not clear how to approximate explicitly an element $\varphi \in W_0^{-1,p}(\Omega)$ by elements of $W_0^{-1,p}(\Omega)$ with compact support.

PROPOSITION 5.1.21. *If $1 < p < \infty$ and $-\triangle$ is defined by*

$$(-\triangle u, \varphi)_{W^{-1,p}, W_0^{1,p'}} = \int_\Omega \nabla u \cdot \nabla \varphi, \tag{5.1.5}$$

*for all $\varphi \in W_0^{1,p'}(\Omega)$, then $-\triangle \in \mathcal{L}(W^{1,p}(\Omega), W^{-1,p}(\Omega))$.*

PROOF. We note that

$$\Bigl| \int_\Omega \nabla u \cdot \nabla \varphi \Bigr| \le \|\nabla u\|_{L^p} \|\nabla \varphi\|_{L^{p'}} \le \|u\|_{W^{1,p}} \|\varphi\|_{W_0^{1,p'}},$$

for all $u \in W_0^{1,p}(\Omega), \varphi \in W_0^{1,p'}(\Omega)$. It follows that (5.1.5) defines an element of $W^{-1,p}(\Omega)$ (note also that this definition is consistent with the classical definition) and that $\| -\triangle u\|_{W^{-1,p}} \le \|u\|_{W^{1,p}}$, i.e. $-\triangle \in \mathcal{L}(W^{1,p}(\Omega), W^{-1,p}(\Omega))$. □

Corollary 5.1.22. *Let*

$$J(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2, \tag{5.1.6}$$

*for* $u \in H^1_0(\Omega)$. *Then* $J \in C^1(H^1_0(\Omega), \mathbb{R})$ *and*

$$J'(u) = -\triangle u, \tag{5.1.7}$$

*for all* $u \in H^1_0(\Omega)$.

Proof. We have

$$J(u+v) - J(u) - (-\triangle u, v)_{H^{-1},H^1_0} = \frac{1}{2}\int_{\Omega} |\nabla v|^2,$$

from which the result follows. □

## 5.2. Sobolev spaces and Fourier transform

When $\Omega = \mathbb{R}^N$, one can characterize the space $W^{m,p}(\mathbb{R}^N)$ in terms of the Fourier transform. For that purpose, it is convenient in this section to consider the Sobolev spaces of complex-valued functions. The case $p = 2$ is especially simple, by using Plancherel's formula. We begin with the following lemma.

Lemma 5.2.1. *Let* $u \in L^2(\mathbb{R}^N)$ *and* $\alpha$ *a multi-index. Then* $D^\alpha u \in L^2(\mathbb{R}^N)$ *if and only if* $|\cdot|^{|\alpha|}\widehat{u} \in L^2(\mathbb{R}^N)$. *Moreover,* $\mathcal{F}(D^\alpha u)(\xi) = (2\pi i)^{|\alpha|}\xi^\alpha \widehat{u}(\xi)$, *where* $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_N}$. *In particular* $\|D^\alpha u\|_{L^2} = (2\pi)^{|\alpha|}\| |\cdot|^{|\alpha|}\widehat{u}\|_{L^2}$.

Proof. Suppose $D^\alpha u \in L^2(\mathbb{R}^N)$, which means that

$$\mathrm{Re}\int_{\mathbb{R}^N} u\overline{D^\alpha\varphi} = (-1)^{|\alpha|}\mathrm{Re}\int_{\mathbb{R}^N} D^\alpha u\overline{\varphi}, \tag{5.2.1}$$

for all $\varphi \in C^\infty_c(\mathbb{R}^N)$. By density, (5.2.1) holds for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$. By Plancherel's formula, (5.2.1) is equivalent to

$$\mathrm{Re}\int_{\mathbb{R}^N} \widehat{u}\overline{\mathcal{F}(D^\alpha\varphi)} = (-1)^{|\alpha|}\mathrm{Re}\int_{\mathbb{R}^N} \mathcal{F}(D^\alpha u)\overline{\widehat{\varphi}}.$$

Since $\mathcal{F}(D^\alpha\varphi) = (2\pi i)^{|\alpha|}\xi^\alpha\widehat{\varphi}$, we deduce that

$$(-2\pi i)^{|\alpha|}\mathrm{Re}\int_{\mathbb{R}^N} \xi^\alpha\widehat{u}\overline{\widehat{\varphi}} = (-1)^{|\alpha|}\mathrm{Re}\int_{\mathbb{R}^N} \mathcal{F}(D^\alpha u)\overline{\widehat{\varphi}},$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$, which means that $\mathcal{F}(D^\alpha u) = (2\pi i)^{|\alpha|}\xi^\alpha\widehat{u}$. Conversely, suppose $|\cdot|^{|\alpha|}\widehat{u} \in L^2(\mathbb{R}^N)$ and let $u_\alpha \in L^2(\mathbb{R}^N)$ be defined by $\widehat{u_\alpha} = (2\pi i)^{|\alpha|}\xi^\alpha\widehat{u}$. Given $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^N} u_\alpha\overline{\varphi} = \mathrm{Re}\int_{\mathbb{R}^N} \widehat{u_\alpha}\overline{\widehat{\varphi}} &= (-1)^{|\alpha|}\mathrm{Re}\int_{\mathbb{R}^N} \widehat{u}\overline{(2\pi i)^{|\alpha|}\xi^\alpha\widehat{\varphi}} \\ &= (-1)^{|\alpha|}\mathrm{Re}\int_{\mathbb{R}^N} \widehat{u}\overline{\mathcal{F}(D^\alpha\varphi)} = (-1)^{|\alpha|}\mathrm{Re}\int_{\mathbb{R}^N} u\overline{D^\alpha\varphi},\end{aligned}$$

so that $D^\alpha u = u_\alpha$. This completes the proof. □

Proposition 5.2.2. *Given any* $m \in \mathbb{Z}$,

$$H^m(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N);\ (1+|\cdot|^2)^{\frac{m}{2}}\widehat{u} \in L^2(\mathbb{R}^N)\},$$

*and* $\|u\|_{H^m} \approx \|(1+|\cdot|^2)^{\frac{m}{2}}\widehat{u}\|_{L^2}$.

Proof. If $m \geq 0$, then the result easily follows from Lemma 5.2.1. Next, if $m \geq 0$, then it is clear that the dual of the space $\{u \in \mathcal{S}'(\mathbb{R}^N);\ (1+|\cdot|^2)^{\frac{m}{2}}\widehat{u} \in L^2(\mathbb{R}^N)\}$ with the norm $\|(1+|\cdot|^2)^{\frac{m}{2}}\widehat{u}\|_{L^2}$ is the space $\{u \in \mathcal{S}'(\mathbb{R}^N);\ (1+|\cdot|^2)^{-\frac{m}{2}}\widehat{u} \in L^2(\mathbb{R}^N)\}$ with the norm $\|(1+|\cdot|^2)^{-\frac{m}{2}}\widehat{u}\|_{L^2}$. The result in the case $m \leq 0$ follows. □

Proposition 5.2.2 can be extended to $W^{m,p}(\mathbb{R}^N)$ with $p \neq 2$. More precisely, we have the following result.

THEOREM 5.2.3. *Given any* $m \in \mathbb{Z}$, $a, b > 0$ *and* $1 < p < \infty$,

$$W^{m,p}(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N);\ \mathcal{F}^{-1}[(a + b|\cdot|^2)^{\frac{m}{2}}\widehat{u}] \in L^p(\mathbb{R}^N)\},$$

*and* $\|u\|_{W^{m,p}} \approx \|\mathcal{F}^{-1}[(a + b|\cdot|^2)^{\frac{m}{2}}\widehat{u}]\|_{L^p}$.

The proof of Theorem 5.2.3 is, as opposed to the proof of Proposition 5.2.2, fairly delicate. It is based on a *Fourier multiplier theorem*, which is a deep result in Fourier analysis. A typical such theorem that can be used is the following. (See Bergh and Löfström [**10**], Theorem 6.1.6, p. 135.)

THEOREM 5.2.4. *Let* $\rho \in L^\infty(\mathbb{R}^N)$ *and let* $\ell > N/2$ *be an integer. Suppose* $\rho \in W^{\ell,\infty}_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ *and*

$$\sup_{|\alpha| \le \ell}\ \operatorname*{ess\,sup}_{\xi \neq 0} |\xi|^{|\alpha|}|\partial^\alpha \rho(\xi)| < \infty.$$

*It follows that for every* $1 < p < \infty$, *there exists a constant* $C_p$ *such that*

$$\|\mathcal{F}^{-1}(\rho\widehat{v})\|_{L^p} \le C_p\|v\|_{L^p}, \tag{5.2.2}$$

*for all* $v \in \mathcal{S}(\mathbb{R}^N)$.

PROOF. We refer the reader to Bergh and Löfström [**10**] for the proof. Note that an essential ingredient in the proof is the Marcinkiewicz interpolation theorem. In fact, only a simplified form of this theorem is needed, namely the form stated in Stein [**41**], §4.2, Theorem 5, p. 21. A very simple proof of this (simplified version of the) Marcinkiewicz interpolation theorem is given in Stein [**41**], pp. 21–22. □

PROOF OF THEOREM 5.2.3. Without loss of generality, we may assume $a = b = 1$. We only prove the result for $m \ge 0$, the case $m < 0$ following easily by duality (see the proof of Proposition 5.2.2). The case $m = 0$ being trivial, we assume $m \ge 1$. We set $V = \{u \in \mathcal{S}'(\mathbb{R}^N);\ \mathcal{F}^{-1}[(1 + |\cdot|^2)^{\frac{m}{2}}\widehat{u}] \in L^p(\mathbb{R}^N)\}$ [1] and $\|u\|_V = \|\mathcal{F}^{-1}[(1 + |\cdot|^2)^{\frac{m}{2}}\widehat{u}]\|_{L^p}$ for all $u \in V$. It is not difficult to show that $(V, \|\cdot\|_V)$ is a Banach space. We now proceed in three steps.

STEP 1. $\mathcal{S}(\mathbb{R}^N)$ is dense in $V$. Let $u \in V$ and set $w = \mathcal{F}^{-1}[(1 + |\cdot|^2)^{\frac{m}{2}}\widehat{u}] \in L^p(\mathbb{R}^N)$. $\mathcal{S}(\mathbb{R}^N)$ being dense in $L^p(\mathbb{R}^N)$, there exists $(w_n)_{n\ge 0} \subset \mathcal{S}(\mathbb{R}^N)$ such that $w_n \to w$ in $L^p(\mathbb{R}^N)$. Setting $u_n = \mathcal{F}^{-1}[(1 + |\cdot|^2)^{-\frac{m}{2}}\widehat{w_n}] \in \mathcal{S}(\mathbb{R}^N)$, this means that $u_n \to u$ in $V$.

STEP 2. $V \hookrightarrow W^{m,p}(\mathbb{R}^N)$. By Step 1, it suffices to show that $\|u\|_{W^{m,p}} \le C\|u\|_V$ for all $u \in \mathcal{S}(\mathbb{R}^N)$. Let $\alpha$ be a multi-index with $|\alpha| \le m$ and let $\rho(\xi) = \xi^\alpha(1 + |\xi|^2)^{-\frac{m}{2}}$. It easily follows that $\rho$ satisfies the assumptions of Theorem 5.2.4. Applying (5.2.2) with $v = \mathcal{F}^{-1}[(1 + |\cdot|^2)^{\frac{m}{2}}\widehat{u}]$, we deduce that $\|\mathcal{F}^{-1}(\xi^\alpha\widehat{u})\|_{L^p} \le C\|u\|_V$. Since $\mathcal{F}^{-1}(\xi^\alpha\widehat{u}) = (2\pi i)^{-|\alpha|}D^\alpha u$, we deduce that $\|D^\alpha u\|_{L^p} \le C\|u\|_V$. The result follows, since $\alpha$ with $|\alpha| \le m$ is arbitrary.

STEP 3. $W^{m,p}(\mathbb{R}^N) \hookrightarrow V$. By density of $\mathcal{S}(\mathbb{R}^N)$ in $W^{m,p}(\mathbb{R}^N)$, it suffices to show that $\|u\|_V \le C\|u\|_{W^{m,p}}$ for all $u \in \mathcal{S}(\mathbb{R}^N)$. Fix a function $\theta \in C^\infty(\mathbb{R})$, $\theta \ge 0$, such that $\theta(t) = 0$ for $|t| \le 1$ and $\theta(t) = 1$ for $|t| \ge 2$. Set

$$\rho(\xi) = (1 + |\xi|^2)^{\frac{m}{2}}\Bigl(1 + \sum_{j=1}^N \theta(\xi_j)|\xi_j|^m\Bigr)^{-1}.$$

[1]Note that $(1 + |\cdot|^2)^{\frac{m}{2}}$ is a $C^\infty$ function with polynomial growth, so that $(1 + |\cdot|^2)^{\frac{m}{2}}\widehat{u}$ is a well-defined element of $\mathcal{S}'(\mathbb{R}^N)$ for all $u \in \mathcal{S}'(\mathbb{R}^N)$.

It is not difficult to show that $\rho$ satisfies the assumptions of Theorem 5.2.4. Applying (5.2.2) with $v = \mathcal{F}^{-1}[(1+\sum_{j=1}^N \theta(\xi_j)|\xi_j|^m)\widehat{u}]$, we deduce that

$$\begin{aligned}\|u\|_V &\le C\Big\|\mathcal{F}^{-1}\Big[\Big(1+\sum_{j=1}^N \theta(\xi_j)|\xi_j|^m\Big)\widehat{u}\Big]\Big\|_{L^p}\\ &\le C\Big(\|u\|_{L^p} + \sum_{j=1}^N \|\mathcal{F}^{-1}(\theta(\xi_j)|\xi_j|^m\widehat{u})\|_{L^p}\Big).\end{aligned} \tag{5.2.3}$$

Next, we observe that $\rho_j(\xi) = \theta(\xi_j)|\xi_j|^m\xi_j^{-m}$ satisfies the assumptions of Theorem 5.2.4. Applying (5.2.2) with $\rho = \rho_j$ and $v = u$, successively for $j = 1, \cdots, N$, we deduce from (5.2.3) that

$$\begin{aligned}\|u\|_V &\le C\Big(\|u\|_{L^p} + \sum_{j=1}^N \|\mathcal{F}^{-1}(\xi_j^m\widehat{u})\|_{L^p}\Big)\\ &= C\Big(\|u\|_{L^p} + (2\pi)^{-m}\sum_{j=1}^N \|\partial_j^m u\|_{L^p}\Big) \le C\|u\|_{W^{m,p}},\end{aligned}$$

which completes the proof. □

## 5.3. The chain rule and applications

We now study the chain rule, and we begin with a simple result.

Proposition 5.3.1. *Let $F \in C^1(\mathbb{R},\mathbb{R})$ satisfy $F(0) = 0$ and $\|F'\|_{L^\infty} = L < \infty$, and consider $1 \le p \le \infty$. If $u \in W^{1,p}(\Omega)$, then $F(u) \in W^{1,p}(\Omega)$ and*

$$\nabla F(u) = F'(u)\nabla u, \tag{5.3.1}$$

*a.e. in $\Omega$. Moreover, if $p < \infty$, then the mapping $u \mapsto F(u)$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. Furthermore, if $p < \infty$ and $u \in W^{1,p}_0(\Omega)$, then $F(u) \in W^{1,p}_0(\Omega)$.*

Proof. We proceed in three steps.

Step 1. The case $u \in C^1_c(\Omega)$. It is immediate that $F(u) \in C^1_c(\Omega)$ and that (5.3.1) holds.

Step 2. The case $u \in W^{1,p}_0(\Omega)$. Suppose $p < \infty$, let $u \in W^{1,p}_0(\Omega)$ and let $(u_n)_{n\ge 0} \subset C^\infty_c(\Omega)$ satisfy $u_n \to u$ in $W^{1,p}_0(\Omega)$ as $n \to \infty$. By possibly extracting a subsequence, we may assume that

$$|u_n| + |\nabla u_n| \le f \in L^p(\Omega),$$

and that

$$u_n \to u, \quad \nabla u_n \to \nabla u,$$

a.e. in $\Omega$. It follows from Step 1 that $F(u_n) \in C^1_c(\Omega) \subset W^{1,p}_0(\Omega)$ and that $\nabla F(u_n) = F'(u_n)\nabla u_n$. In particular,

$$|\nabla F(u_n)| \le L|\nabla u_n| \le Lf.$$

Since $F'(u_n)\nabla u_n \to F'(u)\nabla u$ a.e., we obtain $\nabla F(u_n) \to F'(u)\nabla u$ in $L^p(\Omega)$. Moreover, since $|F(u_n) - F(u)| \le L|u_n - u|$, we have $F(u_n) \to F(u)$ in $L^p(\Omega)$. This implies that $F(u_n) \to F(u)$ in $W^{1,p}_0(\Omega)$ and that (5.3.1) holds.

Step 3. The case $u \in W^{1,p}(\Omega)$. We have $F(u) \in L^p(\Omega)$. Furthermore, given $\varphi \in C^1_c(\Omega)$, let $\xi \in C^1_c(\Omega)$ satisfy $\xi = 1$ on $\operatorname{supp}\varphi$. By Remark 5.1.4 (i) and

Remark 5.1.10 (i), we have $\xi u \in W_0^{1,q}(\Omega)$ for all $1 \le q < \infty$ such that $q \le p$. It follows from Step 2 that

$$\int_\Omega F(u)\nabla\varphi = \int_\Omega F(\xi u)\nabla\varphi = -\int_\Omega \varphi F'(\xi u)\nabla(\xi u) = -\int_\Omega \varphi F'(u)\nabla u.$$

Since clearly $F'(u)\nabla u \in L^p(\Omega)$, we deduce that $F(u) \in W^{1,p}(\Omega)$ and that (5.3.1) holds.

STEP 4. Continuity. Suppose $p < \infty$ and $u_n \to u$ in $W^{1,p}(\Omega)$. We show that $F(u_n) \to F(u)$ in $W^{1,p}(\Omega)$ by contradiction. Thus we assume that $\|F(u_n) - F(u)\|_{W^{1,p}} \ge \varepsilon > 0$. We have $F(u_n) \to F(u)$ in $L^p(\Omega)$. By possibly extracting a subsequence, we may assume that $u_n \to u$ and $\nabla u_n \to \nabla u$ a.e. It follows by dominated convergence that $F'(u_n)\nabla u_n \to F'(u)\nabla u$ in $L^p(\Omega)$. Thus $F(u_n) \to F(u)$ in $W^{1,p}(\Omega)$, which is absurd. □

REMARK 5.3.2. One can prove the following stronger result. If $F : \mathbb{R} \to \mathbb{R}$ is (globally) Lipschitz continuous and if $F(0) = 0$, then for every $u \in W^{1,p}(\Omega)$, we have $F(u) \in W^{1,p}(\Omega)$. Moreover, $\nabla F(u) = F'(u)\nabla u$ a.e. This formula makes sense, since $F'$ exists a.e. and $\nabla u = 0$ a.e. on the set $\{x \in \Omega;\ u(x) \in A\}$ where $A \subset \mathbb{R}$ is any set of measure 0. Furthermore, the mapping $u \to F(u)$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ if $p < \infty$. Finally, if $p < \infty$ and $u \in W_0^{1,p}(\Omega)$, then $F(u) \in W_0^{1,p}(\Omega)$. The proof is rather delicate and makes use in particular of Lebesgue's points theory. (See Marcus and Mizel [35]). We will establish below a particular case of that result.

PROPOSITION 5.3.3. *Set $u^+ = \max\{u, 0\}$ for all $u \in \mathbb{R}$ and let $1 \le p \le \infty$. If $u \in W^{1,p}(\Omega)$, then $u^+ \in W^{1,p}(\Omega)$. Moreover,*

$$\nabla u^+ = \begin{cases} \nabla u \text{ if } u > 0, \\ 0 \text{ if } u \le 0, \end{cases} \tag{5.3.2}$$

*a.e. If $p < \infty$, then the mapping $u \mapsto u^+$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. Furthermore, if $u \in W_0^{1,p}(\Omega)$, then $u^+ \in W_0^{1,p}(\Omega)$*

PROOF. We proceed in four steps.

STEP 1. If $p < \infty$ and $u \in W_0^{1,p}(\Omega)$, then $u^+ \in W_0^{1,p}(\Omega)$ and (5.3.2) holds. Given $\varepsilon > 0$, let

$$\varphi_\varepsilon(u) = \begin{cases} \sqrt{\varepsilon^2 + u^2} - \varepsilon \text{ if } u \ge 0, \\ 0 \text{ if } u \le 0. \end{cases}$$

It follows from Proposition 5.3.1 that $\varphi_\varepsilon(u) \in W_0^{1,p}(\Omega)$ and that $\nabla\varphi_\varepsilon(u) = \varphi_\varepsilon'(u)\nabla u$ a.e. We deduce easily that $\varphi_\varepsilon(u) \to u^+$ and that $\nabla\varphi_\varepsilon(u)$ converges to the right-hand side of (5.3.2) in $L^p(\Omega)$ as $\varepsilon \downarrow 0$. Thus $u^+ \in W_0^{1,p}(\Omega)$ and (5.3.2) holds.

STEP 2. If $u \in W^{1,p}(\Omega)$, then $u^+ \in W^{1,p}(\Omega)$ and (5.3.2) holds. Using Step 1, this is proved by the argument in Step 3 of the proof of Proposition 5.3.1.

STEP 3. If $a \in \mathbb{R}$ and $u \in W^{1,p}(\Omega)$, then $\nabla u = 0$ a.e. on the set $\{x \in \Omega;\ u(x) = a\}$. Consider a function $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta(x) = 1$ for $|x| \le 1$, $\eta(x) = 0$ for $|x| \ge 2$ and $0 \le \eta \le 1$. For $n \in \mathbb{N}$, $n \ge 1$, set

$$g_n(x) = \eta(n(x - a)),$$

and

$$h_n(x) = \int_0^x g_n(s)\,ds.$$

It follows from Proposition 5.3.1 that $h_n(u) \in W^{1,p}(\Omega)$ and that $\nabla h_n(u) = g_n(u)\nabla u$ a.e. Therefore,

$$-\int_{\mathbb{R}} h_n(u)\nabla\varphi = \int_{\mathbb{R}} g_n(u)\varphi\nabla u,$$

for all $\varphi \in C^1_{\rm c}(\Omega)$. Since $|h_n| \le n^{-1}\|\eta\|_{L^1}$, the left-hand side of the above inequality tends to 0 as $n \to \infty$. Therefore,

$$\int_{\mathbb{R}} g_n(u)\varphi\nabla u \underset{n\to\infty}{\longrightarrow} 0.$$

Note that $g_n(u) \to 1_{\{x\in\Omega;\, u(x)=a\}}$. Since $0 \le g_n \le 1$, we deduce that

$$\int_{\mathbb{R}} 1_{\{x\in\Omega;\, u(x)=a\}}\varphi\nabla u = 0,$$

for all $\varphi \in C^1_{\rm c}(\Omega)$; and so, $1_{\{x\in\Omega;\, u(x)=a\}}\nabla u = 0$ a.e. The result follows.

STEP 4. Continuity. Suppose $p < \infty$ and let $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$. We have $|u^+ - u_n^+| \le |u - u_n|$, so that $u_n^+ \to u^+$ in $L^p(\Omega)$. Therefore, we need only show that for any subsequence, which we still denote by $(u_n)_{n\ge0}$, there exists a subsequence $(u_{n_k})_{k\ge0}$ such that $\nabla u_{n_k}^+ \to \nabla u^+$ in $L^p(\Omega)$ as $k \to \infty$. We may extract a subsequence $(u_{n_k})_{k\ge0}$ such that $u_{n_k} \to u$ and $\nabla u_{n_k} \to \nabla u$ a.e., and such that

$$|u_{n_k}| + |\nabla u_{n_k}| \le f \in L^p(\Omega).$$

Set

$$\begin{aligned} &A_0 = \{x \in \Omega;\, u(x) = 0\},\\ &A^+ = \{x \in \Omega;\, u(x) > 0\}, \quad A_k^+ = \{x \in \Omega;\, u_{n_k}(x) > 0\},\\ &A^- = \{x \in \Omega;\, u(x) < 0\}, \quad A_k^- = \{x \in \Omega;\, u_{n_k}(x) < 0\}. \end{aligned}$$

For a.a. $x \in A^+$, we have $x \in A_k^+$ for $k$ large, thus $\nabla u_{n_k}^+(x) = \nabla u_{n_k}(x) \to \nabla u(x) = \nabla u^+(x)$. For a.a. $x \in A^-$, we have $x \in A_k^-$ for $k$ large, hence $\nabla u_{n_k}^+(x) = 0 = \nabla u^+(x)$. For $x \in A_0$, we have $u(x) = 0$, so that by Step 3, $\nabla u(x) = 0$ a.e. Since $\nabla u_{n_k} \to \nabla u = 0$ a.e. on $A_0$, we deduce in particular that $|\nabla u_{n_k}^+| \le |\nabla u_{n_k}| \to 0$ a.e. in $A_0$. Thus $\nabla u_{n_k}^+ \to 0 = \nabla u^+$ a.e. on $A_0$. It follows that

$$\nabla u_{n_k}^+ \to \begin{cases} \nabla u \text{ if } u > 0,\\ 0 \text{ if } u \le 0, \end{cases}$$

a.e., and the result follows by dominated convergence. This completes the proof. □

REMARK 5.3.4. Let $u^- = \max\{0, -u\}$. Since $u^- = (-u)^+$, we may draw similar conclusions for $u^-$. In particular, if $u \in W^{1,p}(\Omega)$, then $u^- \in W^{1,p}(\Omega)$. Moreover,

$$\nabla u^- = \begin{cases} -\nabla u \text{ if } u < 0,\\ 0 \text{ if } u \ge 0, \end{cases}$$

a.e. If $p < \infty$, then the mapping $u \to u^-$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. Furthermore, if $u \in W_0^{1,p}(\Omega)$, then $u^- \in W_0^{1,p}(\Omega)$. Since $|u| = u^+ + u^-$, we deduce the following properties. If $u \in W^{1,p}(\Omega)$, then $|u| \in W^{1,p}(\Omega)$. Moreover,

$$\nabla|u| = \begin{cases} \nabla u \text{ if } u > 0,\\ -\nabla u \text{ if } u < 0,\\ 0 \text{ if } u = 0, \end{cases}$$

a.e. Note in particular that

$$|\nabla|u|\,| = |\nabla u|,$$

a.e. If $p < \infty$, then the mapping $u \to |u|$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. Furthermore, if $u \in W_0^{1,p}(\Omega)$, then $|u| \in W_0^{1,p}(\Omega)$.

COROLLARY 5.3.5. *Let $1 \le p < \infty$, let $u \in W^{1,p}(\Omega)$ and $v \in W_0^{1,p}(\Omega)$. If $|u| \le |v|$ a.e., then $u \in W_0^{1,p}(\Omega)$.*

PROOF. It follows from Remark 5.3.4 that $|v| \in W_0^{1,p}(\Omega)$. Let $(w_n)_{n\geq 0} \subset C_c^\infty(\Omega)$ satisfy $w_n \to |v|$ in $W^{1,p}(\Omega)$ as $n \to \infty$. It follows that $w_n - u^+ \to |v| - u^+$ in $W^{1,p}(\Omega)$, so that $(w_n - u^+)^+ \to (|v| - u^+)^+$ in $W^{1,p}(\Omega)$ by Proposition 5.3.3. Since $(w_n - u^+)^+ \leq w_n^+$, we see that $(w_n - u^+)^+$ has compact support; and so $(w_n - u^+)^+ \in W_0^{1,p}(\Omega)$. We deduce that $(|v| - u^+)^+ \in W_0^{1,p}(\Omega)$. Since $(|v| - u^+)^+ = |v| - u^+$, we see that $u^+ \in W_0^{1,p}(\Omega)$. One shows as well that $u^- \in W_0^{1,p}(\Omega)$, and the result follows. □

COROLLARY 5.3.6. *Let $1 \leq p \leq \infty$ and let $M \geq 0$. If $u \in W^{1,p}(\Omega)$, then $(u - M)^+ \in u \in W^{1,p}(\Omega)$ and*

$$\nabla (u - M)^+ = \begin{cases} \nabla u \text{ if } u(x) > M, \\ 0 \text{ if } u(x) \leq M, \end{cases} \tag{5.3.3}$$

*a.e. in $\Omega$. If $p < \infty$, then the mapping $u \mapsto (u - M)^+$ is continuous $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$. Moreover, if $u \in W_0^{1,p}(\Omega)$, then $(u - M)^+ \in W_0^{1,p}(\Omega)$.*

PROOF. The last property is a consequence of Corollary 5.3.5, since $(u - M)^+ \leq u^+ \in W_0^{1,p}(\Omega)$. Next, observe that if $\Omega$ is bounded, then the conclusions are a consequence of Proposition 5.3.3, because $u - M \in W^{1,p}(\Omega)$ whenever $u \in W^{1,p}(\Omega)$. In particular, we see that for an arbitrary $\Omega$, if $u \in W^{1,p}(\Omega)$, then $(u - M)^+ \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and (5.3.3) holds. In particular, $|\nabla (u - M)^+| \leq |\nabla u| \in L^p(\Omega)$. Since $(u - M)^+ \leq u^+ \in L^p(\Omega)$, we see that $(u - M)^+ \in W^{1,p}(\Omega)$ and that (5.3.3) holds.

It now remains to show the continuity of the mapping $u \mapsto (u - M)^+$ when $p < \infty$. By the above observation, we may assume that $\Omega$ is unbounded. Given $R > 0$, let $\Omega_R = \{x \in \Omega;\ |x| < R\}$ and $U_R = \Omega \setminus \Omega_R$. We argue by contradiction, and we consider a sequence $(u_n)_{n\geq 0} \subset W^{1,p}(\Omega)$ and $u \in W^{1,p}(\Omega)$ such that $u_n \underset{n\to\infty}{\longrightarrow} u$ in $W^{1,p}(\Omega)$ and $\|(u_n - M)^+ - (u - M)^+\|_{W^{1,p}} \geq \varepsilon > 0$. Note that

$$|(u_n - M)^+ - (u - M)^+| \leq |u_n - u| \underset{n\to\infty}{\longrightarrow} 0,$$

in $L^p(\Omega)$, so that we may assume $\|\nabla (u_n - M)^+ - \nabla (u - M)^+\|_{L^p} \geq \varepsilon > 0$. By possibly extracting a subsequence, we may also assume that there exists $f \in L^p(\Omega)$ such that $|\nabla u_n| + |\nabla u| \leq f$ a.e. In particular, it follows from (5.3.3) that $|\nabla (u_n - M)^+ - \nabla (u - M)^+| \leq |\nabla u_n| + |\nabla u| \leq f$ a.e. Therefore, by dominated convergence, we may choose $R$ large enough so that

$$\|\nabla (u_n - M)^+ - \nabla (u - M)^+\|_{L^p(U_R)} \leq \frac{\varepsilon}{4}.$$

Finally, since $\Omega_R$ is bounded, it follows that $\|\nabla (u_n - M)^+ - \nabla (u - M)^+\|_{L^p(\Omega_R)} \to 0$ as $n \to \infty$. Therefore, for $n$ large enough,

$$\|\nabla (u_n - M)^+ - \nabla (u - M)^+\|_{L^p(\Omega_R)} \leq \frac{\varepsilon}{4}.$$

We deduce that $\|\nabla (u_n - M)^+ - \nabla (u - M)^+\|_{L^p(\Omega)} \leq \varepsilon/2$, which yields a contradiction. This completes the proof. □

COROLLARY 5.3.7. *Let $1 \leq p \leq \infty$, $(u_n)_{n\geq 0} \subset W^{1,p}(\Omega)$ and $u \in W^{1,p}(\Omega)$. If $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$, then there exist a subsequence $(u_{n_k})_{k\geq 0}$ and $v \in W^{1,p}(\Omega)$ such that $|u_{n_k}| \leq v$ a.e. in $\Omega$ for all $k \geq 0$. If, in addition, $p < \infty$ and $(u_n)_{n\geq 0} \subset W_0^{1,p}(\Omega)$, then one can choose $v \in W_0^{1,p}(\Omega)$.*

PROOF. Let the subsequence $(u_{n_k})_{k\geq 0}$ satisfy $\|u_{n_k} - u\|_{W^{1,p}} \leq 2^{-k-1}$, so that $\|u_{n_{k+1}} - u_{n_k}\|_{W^{1,p}} \leq 2^{-k}$. It follows from Remark 5.3.4 that $|u_{n_{k+1}} - u_{n_k}| \in W^{1,p}(\Omega)$

and that $\| |u_{n_{k+1}} - u_{n_k}| \|_{W^{1,p}} \le 2^{-k}$. Thus, the series

$$v = |u_{n_0}| + \sum_{j\ge 0} |u_{n_{j+1}} - u_{n_j}|,$$

is normally convergent in $W^{1,p}(\Omega)$. Since

$$u_{n_{k+1}} = u_{n_0} + \sum_{j=0}^{k} (u_{n_{j+1}} - u_{n_j}),$$

we see that $|u_{n_{k+1}}| \le v$. The result follows, using again Remark 5.3.4 in the case $p < \infty$ and $(u_n)_{n\ge 0} \subset W^{1,p}_0(\Omega)$. □

COROLLARY 5.3.8. *Let $1 \le p < \infty$, $0 \le A, B \le \infty$ and set*

$$E = \{u \in W^{1,p}_0(\Omega);\ -A \le u \le B \ a.e.\},$$
$$F = \{u \in C^{\infty}_{\mathrm{c}}(\Omega);\ -A \le u \le B\}.$$

*It follows that $E = \overline{F}$, where the closure is in $W^{1,p}_0(\Omega)$. In particular, $\{u \in W^{1,p}_0(\Omega);\ u \ge 0 \ a.e.\}$ is the closure in $W^{1,p}_0(\Omega)$ of $\{u \in C^{\infty}_{\mathrm{c}}(\Omega);\ u \ge 0\}$.*

PROOF. We have $F \subset E$. Since $E$ is clearly closed in $W^{1,p}_0(\Omega)$, we deduce that $\overline{F} \subset E$. We now show the converse inclusion. Let $u \in E$ and let $(u_n)_{n\ge 0} \subset C^{\infty}_{\mathrm{c}}(\Omega)$ be such that $u_n \to u$ in $W^{1,p}_0(\Omega)$. Set

$$v_n = \max\{-A, \min\{u_n, B\}\} = u_n + (u_n + A)^- - (u_n - B)^+.$$

It follows from Corollary 5.3.6 that $v_n \in W^{1,p}_0(\Omega)$ and that

$$v_n \underset{n\to\infty}{\longrightarrow} u + (u + A)^- - (u - B)^+ = u,$$

in $W^{1,p}_0(\Omega)$. Thus if $(v_n)_{n\ge 0} \subset \overline{F}$, then the conclusion follows. Since clearly $v_n \in C_{\mathrm{c}}(\Omega)$, we need only show the following property: if $w \in E \cap C_{\mathrm{c}}(\Omega)$, then $w \in \overline{F}$. To see this, let $(\rho_n)_{n\ge 0}$ be a smoothing sequence and set $\widetilde{w}_n = \rho_n \star \widetilde{w}$, where $\widetilde{w}$ is the extension of $w$ by 0 outside $\Omega$. Since $w$ has compact support in $\Omega$, we see that if $n$ is sufficiently large, then $\widetilde{w}_n$ also has compact support in $\Omega$. Moreover, $\widetilde{w}_n \in C^{\infty}_{\mathrm{c}}(\Omega)$, so that if $w_n = (\widetilde{w}_n)_{|\Omega}$, then $w_n \in C^{\infty}_{\mathrm{c}}(\Omega)$. In addition, $\widetilde{w}_n \to \widetilde{w}$ in $W^{1,p}(\mathbb{R}^N)$, so that $w_n \to w$ in $W^{1,p}_0(\Omega)$. It remains to show that $-A \le \widetilde{w}_n \le B$, which is immediate since $-A \le \widetilde{w} \le B$. This completes the proof. □

## 5.4. Sobolev's inequalities

In this section, we establish some Sobolev-type inequalities and embeddings. It is convenient to make the following definition.

DEFINITION 5.4.1. Given an integer $m \ge 0$, $1 \le p \le \infty$ and $\Omega$ and open subset of $\mathbb{R}^N$, we set

$$|u|_{m,p,\Omega} = \sum_{|\alpha|=m} \|D^{\alpha}u\|_{L^p(\Omega)}.$$

When there is no risk of confusion, we set

$$|u|_{m,p} = |u|_{m,p,\Omega},$$

i.e. we omit the dependence on $\Omega$.

We begin with inequalities for smooth functions on $\mathbb{R}^N$. The following result is the main inequality of this section.

THEOREM 5.4.2 (Gagliardo-Nirenberg's inequality). *Consider $1 \le p, q, r \le \infty$ and let $j, m$ be two integers, $0 \le j < m$. If*

$$\frac{1}{p} - \frac{j}{N} = a\Big(\frac{1}{r} - \frac{m}{N}\Big) + \frac{1-a}{q}, \tag{5.4.1}$$

*for some $a \in [j/m, 1]$ ($a < 1$ if $r = N/(m-j) > 1$), then there exists a constant $C = C(N, m, j, a, q, r)$ such that*

$$|u|_{j,p} \le C |u|_{m,r}^{a} \|u\|_{L^q}^{1-a}, \tag{5.4.2}$$

*for all $u \in C_c^m(\mathbb{R}^N)$.*

The proof of Theorem 5.4.2 uses various important inequalities. The fundamental ingredients are Sobolev's inequality (Theorem 5.4.5), Morrey's inequality (Theorem 5.4.8), and an inequality for intermediate derivatives (Theorem 5.4.10). We begin with the following first-order Sobolev inequality.

THEOREM 5.4.3. *Let $N \ge 1$. For every $u \in C_c^1(\mathbb{R}^N)$, we have*

$$\|u\|_{L^{\frac{N}{N-1}}} \le \frac{1}{2} \prod_{j=1}^{N} \Big\| \frac{\partial u}{\partial x_j} \Big\|_{L^1}^{\frac{1}{N}}. \tag{5.4.3}$$

*In particular,*

$$\|u\|_{L^{\frac{N}{N-1}}} \le \frac{1}{2N} |u|_{1,1}, \tag{5.4.4}$$

*for all $u \in C_c^1(\mathbb{R}^N)$.*

PROOF. We proceed in three steps.

STEP 1. The case $N = 1$. Given $x \in \mathbb{R}$, we have

$$u(x) = \int_{-\infty}^{x} u'(s)\, ds;$$

and so,

$$|u(x)| \le \int_{-\infty}^{x} |u'(s)|\, ds.$$

As well,

$$|u(x)| \le \int_{x}^{+\infty} |u'(s)|\, ds,$$

so that by summing up the two above inequalities,

$$|u(x)| \le \frac{1}{2} \int_{-\infty}^{+\infty} |u'(s)|\, ds,$$

which yields (5.4.3) (and (5.4.4)) in the case $N = 1$.

STEP 2. Proof of (5.4.3). We assume $N \ge 2$. For any $1 \le j \le N$, it follows from Step 1 that

$$|u(x)| \le \frac{1}{2} \int_{\mathbb{R}} |\partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_N)|\, ds;$$

and so,

$$|u(x)|^N \le 2^{-N} \prod_{j=1}^{N} \int_{\mathbb{R}} |\partial_j u(x_1, \ldots, x_{j-1}, s, x_{j+1}, \ldots, x_N)|\, ds.$$

Taking the $(N-1)^{\text{th}}$ root and integrating on $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N} |u(x)|^{\frac{N}{N-1}}\, dx \le$$

$$2^{-\frac{N}{N-1}} \int_{\mathbb{R}^N} \prod_{j=1}^{N} \Big( \int_{\mathbb{R}} |\partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_N)| \, ds \Big)^{\frac{1}{N-1}}.$$

We observe that the right-hand side is the product of $N$ functions, each of which depends only on $N-1$ of the variables $x_1, \dots, x_N$ (with a permutation). Therefore, integrating in each of the variables $x_1, \dots, x_N$, we may apply Hölder's inequality

$$\int_{\mathbb{R}} a_1^{\frac{1}{N-1}} \cdots a_{N-1}^{\frac{1}{N-1}} \le \prod_{\ell=1}^{N-1} \Big( \int_{\mathbb{R}} a_\ell \Big)^{\frac{1}{N-1}}.$$

For example, if we first integrate in $x_1$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} dx_1 \prod_{j=1}^{N} \Big( \int_{\mathbb{R}} |\partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_N)| \, ds \Big)^{\frac{1}{N-1}} \\ = \Big( \int_{\mathbb{R}} |\partial_1 u(s, x_2, \dots, x_N)| \, ds \Big)^{\frac{1}{N-1}} \\ \times \int_{\mathbb{R}} \prod_{j=2}^{N} \Big( \int_{\mathbb{R}} |\partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_N)| \, ds \Big)^{\frac{1}{N-1}} \\ \le \Big( \int_{\mathbb{R}} |\partial_1 u(s, x_2, \dots, x_N)| \, ds \Big)^{\frac{1}{N-1}} \\ \times \prod_{j=2}^{N} \Big( \int_{\mathbb{R}^2} |\partial_j u(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_N)| \, ds dx_1 \Big)^{\frac{1}{N-1}}. \end{aligned}$$

Integrating successively in each of the variables $x_1, \dots, x_N$, we obtain finally the estimate (5.4.3).

Step 3. Proof of (5.4.4). We claim that if $(a_j)_{1 \le j \le N} \in \mathbb{R}^N$ with $a_j \ge 0$, then

$$\Big( \prod_{j=1}^{N} a_j \Big)^{\frac{1}{N}} \le \frac{1}{N} \sum_{j=1}^{N} a_j. \tag{5.4.5}$$

The estimate (5.4.4) is a consequence of (5.4.3) and (5.4.5). The claim (5.4.5) follows if show that

$$\max_{|x|^2=1} \prod_{j=1}^{N} x_j^2 = N^{-N}. \tag{5.4.6}$$

To prove (5.4.6), we observe that if the maximum is achieved at $x$, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $F'(x) = \lambda x$, where $F(x) = x_1^2 \dots x_N^2$. This implies that

$$2x_i \sum_{j \ne i} x_j^2 = \lambda x_i,$$

for all $1 \le i \le N$. Since none of the $x_i$ vanishes (for the maximum is clearly positive), this implies that $x_1^2 = \dots = x_N^2$, from which (5.4.6) follows. □

Corollary 5.4.4. *Let $1 \le r \le N$ ($r < N$ if $N \ge 2$). If $r^* > r$ is defined by*

$$\frac{1}{r^*} = \frac{1}{r} - \frac{1}{N},$$

*then*

$$\|u\|_{L^{r^*}} \le c_{N,r} |u|_{1,r}, \tag{5.4.7}$$

*for every $u \in C^1_c(\mathbb{R}^N)$, with $c_{N,r} = (N-1)r/2N(N-r)$. (We use the convention that $(N-1)/(N-1) = 1$ if $N = 1$.)*

Proof. The case $N = 1$ follows from Theorem 5.4.3, so we assume $N \geq 2$. Let

$$t = \frac{N-1}{N} r^* = \frac{(N-1)r}{N-r}.$$

Since $r \geq 1$, we have $t \geq 1$. We observe that

$$\frac{Nt}{N-1} = (t-1)r' = r^*,$$

and we apply (5.4.4) with $u$ replaced by $|u|^{t-1}u$, and we obtain

$$\|u\|_{L^{r^*}}^t \leq (2N)^{-1} |\, |u|^{t-1}u|_{1,1}. \tag{5.4.8}$$

It follows from (5.3.1) that $\partial_j(|u|^{t-1}u) = t|u|^{t-1}\partial_j u$ for all $1 \leq j \leq N$. Therefore, by Hölder's inequality,

$$\|\partial_j(|u|^{t-1}u)\|_{L^1} \leq t\|u\|_{L^{(t-1)r'}}^{t-1} \|\partial_j u\|_{L^r} = t\|u\|_{L^{r^*}}^{t-1} \|\partial_j u\|_{L^r}.$$

Thus $|\, |u|^{t-1}u|_{1,1} \leq t\|u\|_{L^{r^*}}^{t-1} |u|_{1,r}$, and we deduce from (5.4.8) that

$$\|u\|_{L^{r^*}}^t \leq (2N)^{-1} t \|u\|_{L^{r^*}}^{t-1} |u|_{1,r}, \tag{5.4.9}$$

and (5.4.7) follows. □

The following Sobolev's inequality is now a consequence of Corollary 5.4.4.

Theorem 5.4.5 (Sobolev's inequality). *Let $m \leq N$ be an integer, let $1 \leq r \leq N/m$ ($r < N/m$ if $N \geq 2$), and let $r^* > r$ be defined by*

$$\frac{1}{r^*} = \frac{1}{r} - \frac{m}{N}.$$

*If*

$$c_{N,m,r} = \frac{[(N-1)r]^m}{(2N)^m \displaystyle\prod_{1 \leq \ell \leq m} (N - \ell r)}, \tag{5.4.10}$$

*then*

$$\|u\|_{L^{r^*}} \leq c_{N,m,r} |u|_{m,r}, \tag{5.4.11}$$

*for all $u \in C_c^m(\mathbb{R}^N)$. (We use the convention that $(N-1)/(N-1) = 1$ if $N = 1$.)*

Proof. We argue by induction on $m$. By Corollary 5.4.4, (5.4.11) holds for $m = 1$. Suppose it holds up to some $m \geq 1$. We suppose that $m + 1 < N$ and we show (5.4.11) at the level $m + 1$. Let $1 \leq r < N/(m+1)$ and let $r^*$ be defined by

$$\frac{1}{r^*} = \frac{1}{r} - \frac{m+1}{N}.$$

Define $p$ by

$$\frac{1}{p} = \frac{1}{r^*} + \frac{1}{N} = \frac{1}{r} - \frac{m}{N}, \tag{5.4.12}$$

so that $r < p < r^*$. It follows from Corollary 5.4.4 and the first identity in (5.4.12) that

$$\|u\|_{L^{r^*}} \leq c_{N,p} |u|_{1,p}.$$

Next, it follows from the second identity in (5.4.12) and (5.4.11) applied to $\partial_j u$ that

$$\|\partial_j u\|_{L^p} \leq c_{N,m,r} |\partial_j u|_{m,r},$$

for all $1 \leq j \leq N$. We deduce that

$$|u|_{1,p} \leq c_{N,m,r} |u|_{m+1,r},$$

and (5.4.11) at the level $m+1$ follows with $c_{N,m+1,r} = c_{N,p} c_{N,m,r}$, i.e. (5.4.10). □

Remark 5.4.6. Note that when $N \geq 2$, the inequality $\|u\|_{L^\infty} \leq C|u|_{1,N}$ does not hold, for any constant $C$. Indeed, given $0 < \theta < 1 - 1/N$, let $f \in C^\infty(0,\infty)$ satisfy $f(r) = |\log r|^\theta$ for $r \leq 1/2$ and $f(r) = 0$ for $r \geq 1$. Let $(f_n)_{n\geq 1} \subset C^\infty([0,\infty))$ be such that $f_n(r) = f(r)$ for $r \geq 1/n$ and $0 \leq f_n(r) \leq f(r)$ and $|f_n'(r)| \leq |f'(r)|$ for all $r > 0$. Setting $u_n(x) = f_n(|x|)$, one verifies easily that $\|u_n\|_{L^\infty} \to \infty$ and $\limsup \|\nabla u_n\|_{L^N} < \infty$ as $n \to \infty$. More generally, a similar example with $0 < \theta < 1 - m/N$ shows that the inequality $\|u\|_{L^\infty} \leq C|u|_{m,N/m}$ does not hold, for any constant $C$ if $1 \leq m < N$.

The following result, in the same spirit as Theorem 5.4.3 (case $N = 1$) shows that the inequality $\|u\|_{L^\infty} \leq C|u|_{N,1}$ holds in any dimension.

Theorem 5.4.7. *Given any $N \geq 1$,*

$$\|u\|_{L^\infty} \leq 2^{-N}|u|_{N,1}, \tag{5.4.13}$$

*for all $u \in C_c^N(\mathbb{R}^N)$.*

Proof. Let $y \in \mathbb{R}^N$ Integrating $\partial_1 \cdots \partial_N u$ in $x_1$ on $(-\infty, y_1)$ yields

$$\partial_2 \cdots \partial_N u(y_1, x_2, \dots, x_N) = \int_{-\infty}^{y_1} \partial_1 \cdots \partial_N u \, dx_1.$$

Integrating successively in the variables $x_2, \dots, x_N$, we obtain

$$u(y) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_N} \partial_1 \cdots \partial_N u \, dx_1 \cdots dx_N.$$

Therefore,

$$|u(y)| \leq \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_N} |\partial_1 \cdots \partial_N u| \, dx_1 \cdots dx_N. \tag{5.4.14}$$

We observe that instead on integrating in $x_1$ on $(-\infty, y_1)$, we might have integrated on $(y_1, \infty)$, thus obtaining

$$|u(y)| \leq \int_{y_1}^{\infty} \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_N} |\partial_1 \cdots \partial_N u| \, dx_1 \cdots dx_N. \tag{5.4.15}$$

Summing up (5.4.14) and (5.4.15), we obtain

$$|u(y)| \leq (1/2) \int_{-\infty}^{\infty} \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_N} |\partial_1 \cdots \partial_N u| \, dx_1 \cdots dx_N.$$

Rereating this argument for each of the variables, we deduce that

$$|u(y)| \leq 2^{-N} \int_{\mathbb{R}^N} |\partial_1 \cdots \partial_N u| \leq |u|_{N,1},$$

and the result follows since $y$ is arbitrary. □

In the case $p > N$, we have the following result.

Theorem 5.4.8 (Morrey's inequality). *If $r > N \geq 1$, then there exists a constant $c(N)$ such that*

$$|u(x) - u(y)| \leq c(N) \frac{r}{r - N} |x - y|^{1 - \frac{N}{r}} |u|_{1,r}, \tag{5.4.16}$$

*for all $u \in C_c^1(\mathbb{R}^N)$. Moreover, if $1 \leq q \leq \infty$ and $a \in [0,1)$ is defined by*

$$0 = a\Big(\frac{1}{r} - \frac{1}{N}\Big) + \frac{1-a}{q},$$

*then*

$$\|u\|_{L^\infty} \leq c(N) \frac{r}{r - N} |u|_{1,r}^a \|u\|_{L^q}^{1-a}, \tag{5.4.17}$$

*for all $u \in C_c^1(\mathbb{R}^N)$.*

PROOF. In the following calculations, we denote by $c(N)$ various constants that may change from line to line but depend only on $N$. Let $z \in \mathbb{R}^N$ and $\rho > 0$, and set $B = B(z, \rho)$. Consider $x \in B$, and assume for simplicity $x = 0$. We have

$$u(y) - u(0) = \int_0^1 \frac{d}{dt} u(ty)\, dt = \int_0^1 y \cdot \nabla u(ty)\, dt,$$

for all $y \in B$. Integrating on $B$ and dividing by $|B|$, we obtain

$$\frac{1}{|B|} \int_B u(y)\, dy - u(0) = \frac{1}{|B|} \int_0^1 \int_B y \cdot \nabla u(ty)\, dy\, dt.$$

Since

$$\begin{aligned}\Big| \int_B y \cdot \nabla u(ty)\, dy \Big| &\le \Big( \int_B |y|^{r'}\, dy \Big)^{\frac{1}{r'}} \Big( \int_B |\nabla u(ty)|^r\, dy \Big)^{\frac{1}{r}} \\ &= (N + r')^{-\frac{1}{r'}} \gamma_N^{\frac{1}{r'}} \rho^{1+\frac{N}{r'}} t^{-\frac{N}{r}} \Big( \int_{tB} |\nabla u(y)|^r\, dy \Big)^{\frac{1}{r}} \\ &\le (N + r')^{-\frac{1}{r'}} \gamma_N^{\frac{1}{r'}} \rho^{1+\frac{N}{r'}} t^{-\frac{N}{r}} \|\nabla u\|_{L^r},\end{aligned}$$

where $\gamma_N$ is the measure of the unit sphere, we deduce

$$\Big| \frac{1}{|B|} \int_B u(y)\, dy - u(0) \Big| \le \frac{Nr}{N - r} (N + r')^{-\frac{1}{r'}} \gamma_N^{-\frac{1}{r}} \rho^{1-\frac{N}{r}} \|\nabla u\|_{L^r}.$$

Since $(N + r')^{-\frac{1}{r'}} \gamma_N^{-\frac{1}{r}}$ is bounded uniformly in $r > 1$, it follows that if $B = B(z, \rho)$ and $x \in B$, then

$$\Big| \frac{1}{|B|} \int_B u(y)\, dy - u(x) \Big| \le c(N) \frac{r}{r - N} \rho^{1-\frac{N}{r}} |u|_{1,r}. \tag{5.4.18}$$

Let now $x_1, x_2 \in \mathbb{R}^N$, $x_1 \neq x_2$ and let $z = (x_1 + x_2)/2$ and $\rho = |x_1 - x_2|$. Applying (5.4.18) successively with $x = x_1$ and $x = x_2$ and making the sum, we obtain

$$|u(x_1) - u(x_2)| \le c(N) \frac{r}{r - N} |x_1 - x_2|^{1-\frac{N}{r}} \|\nabla u\|_{L^r},$$

which proves (5.4.16).

Consider now $1 \le q \le \infty$. We have

$$\Big| \int_B u(y)\, dy \Big| \le |B|^{\frac{1}{q'}} \|u\|_{L^q};$$

and so,

$$\begin{aligned}\frac{1}{|B|} \Big| \int_B u(y)\, dy \Big| &\le |B|^{-\frac{1}{q}} \|u\|_{L^q} = N^{\frac{1}{q}} \gamma_N^{-\frac{1}{q}} \rho^{-\frac{N}{q}} \|u\|_{L^q} \\ &\le c(N) \rho^{-\frac{N}{q}} \|u\|_{L^q}.\end{aligned}$$

Therefore, we deduce from (5.4.18) that

$$|u(x)| \le c(N) \rho^{-\frac{N}{q}} \|u\|_{L^q} + c(N) \frac{r}{r - N} \rho^{1-\frac{N}{r}} \|\nabla u\|_{L^r}.$$

We now choose $\rho = \|u\|_{L^q}^{\alpha} \|\nabla u\|_{L^r}^{-\alpha}$, with $1 = \alpha(1 - N/r + N/q)$, and we obtain

$$|u(x)| \le c(N) \frac{r}{r - N} \|\nabla u\|_{L^r}^{a} \|u\|_{L^q}^{1-a}.$$

Since $x \in \mathbb{R}^N$ is arbitrary, this proves (5.4.17). □

For the proof of Theorem 5.4.2, we will use the following (first-order) Gagliardo-Nirenberg's inequality, which is a consequence of Sobolev and Morrey's inequalities.

THEOREM 5.4.9. *Let $1 \le p, q, r \le \infty$ and assume*

$$\frac{1}{p} = a\Big(\frac{1}{r} - \frac{1}{N}\Big) + \frac{1-a}{q}, \tag{5.4.19}$$

*for some $a \in [0,1]$ ($a < 1$ if $r = N \ge 2$). It follows that there exists a constant $C = C(N,p,q,r,a)$ such that*

$$\|u\|_{L^p} \le C|u|_{1,r}^a \|u\|_{L^q}^{1-a}, \tag{5.4.20}$$

*for every $u \in C_c^1(\mathbb{R}^N)$.*

PROOF. We consider separately several cases.

THE CASE $r > N$. Note that in this case, $p \ge q$, so that by Hölder's inequality,

$$\|u\|_{L^p} \le \|u\|_{L^\infty}^{\frac{p-q}{p}} \|u\|_{L^q}^{\frac{q}{p}}.$$

Estimating $\|u\|_{L^\infty}$ by (5.4.17), we deduce (5.4.20).

THE CASE $r < N$ (THUS $N \ge 2$). Let $r^* = Nr/(N-r)$. It follows from Hölder's inequality that

$$\|u\|_{L^p} \le \|u\|_{L^{r^*}}^a \|u\|_{L^q}^{1-a},$$

with $a$ given by (5.4.19). (5.4.20) follows, estimating $\|u\|_{L^{r^*}}$ by (5.4.7).

THE CASE $r = N$. Suppose first $N = 1$. Then by Hölder's inequality,

$$\|u\|_{L^p} \le \|u\|_{L^\infty}^a \|u\|_{L^q}^{1-a},$$

and the result follows from (5.4.4). In the case $N \ge 2$ (thus $a < 1$) we cannot use the same argument since $\|u\|_{L^\infty}$ is *not* estimated in terms of $\|\nabla u\|_{L^N}$. Instead, we apply (5.4.4) with $u$ replaced by $|u|^{t-1}u$ for some $t \ge 1$. As in the proof of (5.4.9), we obtain

$$\|u\|_{L^{\frac{tN}{N-1}}}^t \le (2N)^{-1} t \|u\|_{L^{\frac{(t-1)N}{N-1}}}^{t-1} |u|_{1,r}. \tag{5.4.21}$$

Suppose first that $p \ge q + N/(N-1)$, and let $t \ge 1$ be defined by

$$\frac{tN}{N-1} = p.$$

It follows that $(t-1)N/(N-1) \ge q$. By Hölder's inequality,

$$\|u\|_{L^{\frac{(t-1)N}{N-1}}} \le \|u\|_{L^p}^\alpha \|u\|_{L^q}^{1-\alpha}, \tag{5.4.22}$$

with

$$\frac{N-1}{(t-1)N} = \frac{\alpha(N-1)}{tN} + \frac{1-\alpha}{q}.$$

It follows from (5.4.21)-(5.4.22) that

$$\|u\|_{L^p}^t \le (2N)^{-1} t \|u\|_{L^p}^{(t-1)\alpha} \|u\|_{L^q}^{(t-1)(1-\alpha)} |u|_{1,r};$$

and so,

$$\|u\|_{L^p} \le (t/2N)^{\frac{1}{t-(t-1)\alpha}} \|u\|_{L^q}^{\frac{(t-1)(1-\alpha)}{t-(t-1)\alpha}} |u|_{1,r}^{\frac{1}{t-(t-1)\alpha}}.$$

Since one verifies easily that

$$\frac{1}{t-(t-1)\alpha} = \frac{p-q}{p} = a, \quad \frac{(t-1)(1-\alpha)}{t-(t-1)\alpha} = \frac{q}{p} = 1-a,$$

this yields (5.4.20), since $t/2N \le p$. For $p < q + N/(N-1)$, we apply Hölder's inequality

$$\|u\|_{L^p} \le \|u\|_{L^{3q}}^{\frac{3(p-q)}{2p}} \|u\|_{L^q}^{\frac{3q-p}{2p}}.$$

(Note that $3q \ge q + 2 \ge p$.) We estimate $\|u\|_{L^{3q}}$ by applying (5.4.20) with $p = 3q$, and the result follows. □

We now study interpolation inequalities for intermediate derivatives.

THEOREM 5.4.10. *Given an integer $m \ge 1$, there exists a constant $C_m$ with the following property. If $0 \le j \le m$ and if $1 \le p, q, r \le \infty$ satisfy*

$$\frac{m}{p} = \frac{j}{r} + \frac{m-1}{q}, \tag{5.4.23}$$

*then for every $i \in \{1, \dots, N\}$,*

$$\|\partial_i^j u\|_{L^p} \le C_m \|u\|_{L^q}^{\frac{m-j}{m}} \|\partial_i^m u\|_{L^r}^{\frac{j}{m}}, \tag{5.4.24}$$

*for all $u \in C_c^m(\mathbb{R}^N)$. Moreover,*

$$|u|_{j,p} \le C_m \|u\|_{L^q}^{\frac{m-j}{m}} |u|_{m,r}^{\frac{j}{m}}, \tag{5.4.25}$$

*for all $u \in C_c^m(\mathbb{R}^N)$.*

The proof of Theorem 5.4.10 is based on the following lemma.

LEMMA 5.4.11. *If $1 \le p, q, r \le \infty$ satisfy*

$$\frac{2}{p} = \frac{1}{q} + \frac{1}{r}, \tag{5.4.26}$$

*then*

$$\|u'\|_{L^p} \le 8 \|u\|_{L^q}^{\frac{1}{2}} \|u''\|_{L^r}^{\frac{1}{2}}, \tag{5.4.27}$$

*for all $u \in C_c^2(\mathbb{R})$.*

PROOF. We first observe that we need only prove (5.4.27) for $r > 1$ and $p < \infty$, since the general case can then be obtained by letting $p \uparrow \infty$ or $r \downarrow 1$. Thus we now assume $p < \infty$ and $r > 1$. Let $0 \le \gamma \le 2$ be defined by

$$\gamma = 1 + \frac{1}{p} - \frac{1}{r}. \tag{5.4.28}$$

so that by (5.4.26),

$$-\gamma = -1 - \frac{1}{q} + \frac{1}{p}. \tag{5.4.29}$$

We observe that $p \le 2r$ by (5.4.26), so that

$$\gamma \ge 1/2. \tag{5.4.30}$$

We now fix $u \in C_c^2(\mathbb{R})$ and, given any interval $I \subset \mathbb{R}$, we set

$$f(I) = |I|^{-\gamma p} \|u\|_{L^q(I)}^p, \tag{5.4.31}$$

$$g(I) = |I|^{\gamma p} \|u''\|_{L^r(I)}^p. \tag{5.4.32}$$

We now proceed in five steps.

STEP 1. The estimate

$$\|v'\|_{L^p(0,1)} \le 4\|v\|_{L^q(0,1)} + 2\|v''\|_{L^r(0,1)}, \tag{5.4.33}$$

holds for all $v \in C^2([0,1])$. Let $\xi(x) = 1 - 2x^2$ for $0 \le x \le 1/2$, $\xi(x) = 2(1-x)^2$ for $1/2 \le x \le 1$. It follows that $\xi \in C^1([0,1]) \cap C^2([0,1] \setminus \{1/2\})$ and $0 \le \xi \le 1$, $\xi(0) = 1$, $\xi'(0) = \xi(1) = \xi'(1) = 0$. Moreover, $\xi''(x) = -4$ for $0 \le x \le 1/2$, $\xi''(x) = 4$ for $1/2 \le x \le 1$. An integration by parts yields

$$\int_0^1 \xi v'' = -v'(0) - 4\int_0^{1/2} v + 4\int_{1/2}^1 v;$$

and so,

$$|v'(0)| \le 4\|v\|_{L^1(0,1)} + \|v''\|_{L^1(0,1)}.$$

Given $0 \le x \le 1$, we deduce that

$$|v'(x)| \le |v'(0)| + \int_0^x |v''| \le 4\|v\|_{L^1(0,1)} + 2\|v''\|_{L^1(0,1)}.$$

$x \in [0,1]$ being arbitrary, we conclude that

$$\|v'\|_{L^\infty(0,1)} \le 4\|v\|_{L^1(0,1)} + 2\|v''\|_{L^1(0,1)}.$$

The estimate (5.4.33) follows by Hölder's inequality.

STEP 2. The estimate

$$\|u'\|_{L^p(a,b)} \le 4(b-a)^{-\gamma}\|u\|_{L^q(a,b)} + 2(b-a)^{\gamma}\|u''\|_{L^r(a,b)}, \tag{5.4.34}$$

holds for all $-\infty < a < b < \infty$. Set $v(x) = u(a + (b-a)x)$, so that $v \in C^2([0,1])$. The estimate (5.4.34) follows by applying (5.4.33) to $v$ then using (5.4.28)-(5.4.29).

STEP 3. If $f$ and $g$ are defined by (5.4.31)-(5.4.32), then the estimate

$$\int_I |u'|^p \le 2^{2p-1}(2^p f(I) + g(I)), \tag{5.4.35}$$

holds for all finite interval $I \subset \mathbb{R}$. This follows from (5.4.34) and the elementary inequality $(x+y)^p \le 2^{p-1}(x^p + y^p)$.

STEP 4. Given any $\delta > 0$, there exist a positive integer $\ell$ and disjoints intervals $I_1, \dots, I_\ell$ such that $\cup_{1\le j\le \ell} \overline{I_j} \supset \operatorname{supp} u$ and with the following properties.

$$\ell \le 1 + |\operatorname{supp} u|/\delta, \tag{5.4.36}$$

$$\begin{cases} \text{either } |I_j| = \delta \text{ and } f(I_j) \le g(I_j) \\ \text{or else } |I_j| > \delta \text{ and } f(I_j) = g(I_j), \end{cases} \tag{5.4.37}$$

for all $1 \le j \le \ell$. Indeed, set $x_0 = \inf \operatorname{supp} u$ and let $I = (x_0, x_0 + \delta)$. If $f(I) \le g(I)$, we let $I_1 = I$. If $f(I) > g(I)$, we observe that the functions $\varphi(t) = f(x_0, x_0 + \delta + t)$, $\psi(t) = g(x_0, x_0 + \delta + t)$ satisfy $\varphi(0) > \psi(0)$ and $\varphi(t) \to 0$, $\psi(t) \to \infty$ as $t \to \infty$ (we use (5.4.30)). Thus there exists $t > 0$ such that $\varphi(t) = \psi(t)$ and we let $I_1 = (x_0, x_0 + \delta + t)$. We then see that $I_1$ satisfies (5.4.37). If $\operatorname{supp} u \not\subset I_1$, we can repeat this construction. Since $\operatorname{supp} u$ is compact and $|I_j| \ge \delta$, we obtain in a finite number of steps, say $\ell$, a collection of disjoint open intervals $I_j$ that all satisfy (5.4.37) and such that

$$\bigcup_{1\le j\le \ell-1} I_j \subset \operatorname{supp} u \subset \bigcup_{1\le j\le \ell} \overline{I_j},$$

which clearly imply (5.4.36).

STEP 5. Conclusion. Fix $\delta > 0$. It follows from Step 4 and (5.4.35) that

$$\int_{\mathbb{R}} |u'|^p \le 2^{2p-1} \sum_{j=1}^{\ell} [2^p f(I_j) + g(I_j)], \tag{5.4.38}$$

We let

$$A_1 = \{j \in \{1, \cdots, \ell\};\ |I_j| = \delta\},$$
$$A_2 = \{j \in \{1, \cdots, \ell\};\ |I_j| > \delta\},$$

so that by (5.4.37)

$$\{1, \cdots, \ell\} = A_1 \cup A_2. \tag{5.4.39}$$

If $j \in A_1$, then $f(I_j) \le g(I_j)$ by (5.4.37), so that

$$\begin{aligned} 2^p f(I_j) + g(I_j) &\le (2^p + 1) g(I_j) \le (2^p + 1)|I_j|^{\gamma p} \|u''\|^p_{L^r(I)} \\ &\le (2^p + 1)\delta^{\gamma p} \|u''\|^p_{L^r(\mathbb{R})}, \end{aligned} \tag{5.4.40}$$

where we applied (5.4.32). We deduce from (5.4.40) and (5.4.36) that

$$\sum_{j\in A_1} [2^p f(I_j) + g(I_j)] \le (2^p + 1)(1 + |\operatorname{supp} u|/\delta)\delta^{\gamma p}\|u''\|^p_{L^r(\mathbb{R})}. \tag{5.4.41}$$

If $j \in A_2$, then $f(I_j) = g(I_j)$ by (5.4.37). Since

$$f(I_j) g(I_j) = \|u\|^p_{L^q(I_j)} \|u''\|^p_{L^r(I_j)},$$

we see that

$$f(I_j) = g(I_j) = \|u\|_{L^q(I_j)}^{\frac{p}{2}} \|u''\|_{L^r(I_j)}^{\frac{p}{2}}, \tag{5.4.42}$$

for all $j \in A_2$. It follows from (5.4.42) that

$$\sum_{j\in A_2} [2^p f(I_j) + g(I_j)] \le (2^p + 1) \sum_{j\in A_2} \|u\|_{L^q(I_j)}^{\frac{p}{2}} \|u''\|_{L^r(I_j)}^{\frac{p}{2}}. \tag{5.4.43}$$

Using (5.4.26) and applying Hölder's inequality for the sum in the right-hand side of (5.4.43), we deduce that

$$\sum_{j\in A_2} [2^p f(I_j) + g(I_j)] \le (2^p + 1)\Big(\sum_{j\in A_2} \|u\|_{L^q(I_j)}^q\Big)^{\frac{p}{2q}} \Big(\sum_{j\in A_2} \|u''\|_{L^r(I_j)}^r\Big)^{\frac{p}{2r}},$$

which implies

$$\sum_{j\in A_2} [2^p f(I_j) + g(I_j)] \le (2^p + 1)\|u\|_{L^q(\mathbb{R})}^{\frac{p}{2}} \|u''\|_{L^r(\mathbb{R})}^{\frac{p}{2}}. \tag{5.4.44}$$

We now deduce from (5.4.38), (5.4.39), (5.4.41) and (5.4.44) that

$$\int_{\mathbb{R}} |u'|^p \le 2^{2p-1}(2^p + 1)\times$$
$$[\|u\|_{L^q(\mathbb{R})}^{\frac{p}{2}} \|u''\|_{L^r(\mathbb{R})}^{\frac{p}{2}} + (1 + |\operatorname{supp} u|/\delta)\delta^{\gamma p}\|u''\|_{L^r(\mathbb{R})}^p]. \tag{5.4.45}$$

Note that by (5.4.28)

$$\gamma p = 1 + p - \frac{p}{r} > 1,$$

since $r > 1$. Letting $\delta \downarrow 0$ in (5.4.45) we obtain

$$\int_{\mathbb{R}} |u'|^p \le 2^{2p-1}(2^p + 1)\|u\|_{L^q(\mathbb{R})}^{\frac{p}{2}} \|u''\|_{L^r(\mathbb{R})}^{\frac{p}{2}}.$$

Since $2^p + 1 \le 2^{p+1}$, we see that $2^{2p-1}(2^p + 1) \le 2^{3p}$ and the estimate (5.4.27) follows by taking the $p^{th}$ root of the above inequality. □

Remark 5.4.12. The proof of Lemma 5.4.11 is fairly technical. Note, however, that some special cases of the inequality (5.4.27) can be established very easily. For example, if $p = q = r$, then setting $f = -u'' + u$, we see that $u = (1/2)e^{-|\cdot|} \star f$, so that $u' = \phi \star f$, with $\phi(x) = (x/2|x|)e^{-|x|}$. By Young's inequality, $\|u'\|_{L^p} \le \|\phi\|_{L^1}\|f\|_{L^p} = \|f\|_{L^p}$. Since $\|f\|_{L^p} \le \|u''\|_{L^p} + \|u\|_{L^p}$, (5.4.27) follows. Another easy case is $p = 2$ (so that $r = q'$). Indeed, $u'^2 = (uu')' - uu''$, so that

$$\int u'^2 = -\int uu'' \le \|u''\|_{L^r}\|u\|_{L^q},$$

by Hölder's inequality, which shows (5.4.27). Note that in both these simple cases, one obtains (5.4.27) with the (better) constant 1.

Proof of Theorem 5.4.10. The cases $j = 0$ and $j = m$ being trivial, we assume $1 \le j \le m - 1$ and we proceed in four steps.

Step 1. If $1 \le p, q, r \le \infty$ satisfy (5.4.26) and $i \in \{1, \dots, N\}$, then

$$\|\partial_i u\|_{L^p(\mathbb{R}^N)} \le 8\|u\|_{L^q(\mathbb{R}^N)}^{\frac{1}{2}} \|\partial_i^2 u\|_{L^r(\mathbb{R}^N)}^{\frac{1}{2}}, \tag{5.4.46}$$

for all $u \in C_c^2(\mathbb{R}^N)$. Indeed, assume first $p < \infty$ and let $x = (x_1, \dots, x_N) \in \mathbb{R}^N$. We apply (5.4.27) to the function

$$v(t) = u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_N),$$

and we deduce that

$$\int_{\mathbb{R}} |v'(t)|^p \le 8^p\Big(\int_{\mathbb{R}} |v(t)|^q\Big)^{\frac{p}{2q}} \Big(\int_{\mathbb{R}} |v''(t)|^r\Big)^{\frac{p}{2r}}.$$

Integrating on $\mathbb{R}^{N-1}$ in the variables $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)$ and applying Hölder's inequality to the right-hand side (note that $2q/p+2r/p=1$), we deduce (5.4.46). The case $p=\infty$ follows by letting $p\uparrow\infty$ in (5.4.46).

STEP 2. If $m\ge 2$ and if $1\le p,q,r\le\infty$ satisfy

$$\frac{m}{p}=\frac{1}{r}+\frac{m-1}{q}, \tag{5.4.47}$$

and $i\in\{1,\dots,N\}$, then

$$\|\partial_i u\|_{L^p}\le 8^{2m-3}\|u\|_{L^q}^{\frac{m-1}{m}}\|\partial_i^m u\|_{L^r}^{\frac{1}{m}}, \tag{5.4.48}$$

for all $u\in C_{\mathrm{c}}^m(\mathbb{R}^N)$. We argue by induction on $m$. By Step 1, (5.4.48) holds for $m=2$. Suppose it holds up to some $m\ge 2$. Assume

$$\frac{m+1}{p}=\frac{1}{r}+\frac{m}{q}, \tag{5.4.49}$$

and let $t$ be defined by

$$\frac{m}{t}=\frac{1}{r}+\frac{m-1}{p}. \tag{5.4.50}$$

In particular, $\min\{p,r\}\le t\le\max\{p,r\}$, so that $1\le t\le\infty$. Applying (5.4.48) to $\partial_i u$, we obtain

$$\|\partial_i^2 u\|_{L^t}\le 8^{2m-3}\|\partial_i^{m+1}u\|_{L^r}^{\frac{1}{m}}\|\partial_i u\|_{L^p}^{\frac{m-1}{m}}. \tag{5.4.51}$$

Now, we observe that by (5.4.49) and (5.4.50), $2/p=1/q+1/t$, so it follows from (5.4.46) that

$$\|\partial_i u\|_{L^p}\le 8\|\partial_i^2 u\|_{L^t}^{\frac{1}{2}}\|u\|_{L^q}^{\frac{1}{2}}. \tag{5.4.52}$$

(5.4.51) and (5.4.52) now yield (5.4.48) at the level $m+1$.

STEP 3. Proof of (5.4.24). We argue by induction on $m\ge 2$. For $m=2$, the result follows from Step 1. Suppose now that up to some $m\ge 2$, (5.4.24) holds for all $1\le j\le m-1$. Assume $1\le j\le m$,

$$\frac{m+1}{p}=\frac{j}{r}+\frac{m+1-j}{q}, \tag{5.4.53}$$

and let $t$ be defined by

$$\frac{m}{p}=\frac{j-1}{r}+\frac{m+1-j}{t}. \tag{5.4.54}$$

We first note that by (5.4.54) and (5.4.53),

$$\begin{aligned}\frac{m+1-j}{t}&=\frac{m}{m+1}\frac{m+1}{p}-\frac{j-1}{r}\\&=\frac{m}{m+1}\frac{m+1-j}{q}+\frac{m+1-j}{(m+1)r}\ge 0,\end{aligned}$$

so that $0\le t\le\infty$. Also, by the above identity, and since $q,r\ge 1$,

$$\begin{aligned}\frac{m+1-j}{t}&=\frac{m}{m+1}\frac{m+1-j}{q}+\frac{m+1-j}{(m+1)r}\\&\le\frac{m(m+1-j)}{m+1}+\frac{m+1-j}{m+1}=m+1-j,\end{aligned}$$

so that $t\ge 1$. Applying (5.4.24) (with $j$ replaced by $j-1$) to $\partial_i u$, we obtain

$$\|\partial_i^j u\|_{L^p}\le C_m\|\partial_i^{m+1}u\|_{L^r}^{\frac{j-1}{m}}\|\partial_i u\|_{L^t}^{\frac{m-j+1}{m}}. \tag{5.4.55}$$

Now, we observe that by (5.4.53) and (5.4.54), $(m+1)/t=1/r+m/q$, so it follows from (5.4.48) (applied with $m$ replaced by $m+1$) that

$$\|\partial_i u\|_{L^t}\le 8^{2m-1}\|\partial_i^{m+1}u\|_{L^r}^{\frac{1}{m+1}}\|u\|_{L^q}^{\frac{m}{m+1}}. \tag{5.4.56}$$

(5.4.55) and (5.4.56) now yield (5.4.24) at the level $m+1$.

STEP 4. Proof of (5.4.25). We note that by (5.4.48),

$$|u|_{1,p} \le 8^{2m-3} \|u\|_{L^q}^{\frac{m-1}{m}} |u|_{m,r}^{\frac{1}{m}},$$

whenever (5.4.47) holds. The proof is now parallel to the proof of the estimate (5.4.24) in Step 3 above. □

PROOF OF THEOREM 5.4.2. We consider several cases, and we proceed in three steps.

STEP 1. The case $(m-j)r < N$. Let $t$ be defined by

$$\frac{1}{t} = \frac{1}{r} - \frac{m-j}{N}, \tag{5.4.57}$$

so that $r < t < \infty$. It follows from Sobolev's inequality (5.4.11) applied to $j^{th}$ derivatives of $u$ that

$$|u|_{j,t} \le C|u|_{m,r}. \tag{5.4.58}$$

Next, let $s$ be defined by

$$\frac{m}{s} = \frac{j}{r} + \frac{m-1}{q}, \tag{5.4.59}$$

so that $\min\{q,r\} \le s \le \max\{q,r\}$. It follows from the interpolation inequality (5.4.25) that

$$|u|_{j,s} \le C\|u\|_{L^q}^{\frac{m-j}{m}} |u|_{m,r}^{\frac{j}{m}}. \tag{5.4.60}$$

It follows from (5.4.1), (5.4.57) and (5.4.59) that

$$\frac{1}{p} = \frac{\theta}{t} + \frac{1-\theta}{s},$$

with

$$\theta = \frac{ma-j}{m-j}.$$

Since $j/m \le a \le 1$, we see that $0 \le \theta \le 1$, and we deduce from Hölder's inequality that

$$|u|_{j,p} \le |u|_{j,t}^{\theta} |u|_{j,s}^{1-\theta} \le C|u|_{m,r}^{\theta} (\|u\|_{L^q}^{\frac{m-j}{m}} |u|_{m,r}^{\frac{j}{m}})^{1-\theta},$$

where we used (5.4.58) and (5.4.60). The estimate (5.4.2) follows.

STEP 2. The case $(m-j)r \ge N$ and $a = 1$. Note that if $a = 1$, then by (5.4.1),

$$\frac{1}{p} = \frac{1}{r} - \frac{m-j}{N} \le 0.$$

The only possibility is $(m-j)r = N$ and $p = \infty$. This is allowed only if $r = 1$, and the result is then a consequence of Theorem 5.4.7.

STEP 3. The case $(m-j)r \ge N$ and $a < 1$. Let $t$ be defined by

$$\frac{m}{t} = \frac{j}{r} + \frac{m-1}{q}, \tag{5.4.61}$$

so that $\min\{q,r\} \le t \le \max\{q,r\}$. It follows from the interpolation inequality (5.4.25) that

$$|u|_{j,t} \le C\|u\|_{L^q}^{\frac{m-j}{m}} |u|_{m,r}^{\frac{j}{m}}. \tag{5.4.62}$$

Next, let $s$ be defined by

$$\frac{m-j}{s} = \frac{1}{r} + \frac{m-j-1}{p}, \tag{5.4.63}$$

so that $\min\{p,r\} \le s \le \max\{p,r\}$. It follows from the interpolation inequality (5.4.25) applied to $j^{th}$ order derivatives of $u$ that

$$|u|_{j+1,s} \le C|u|_{m,r}^{\frac{1}{m-j}} |u|_{j,p}^{\frac{m-j-1}{m-j}}. \tag{5.4.64}$$

Next, let $\alpha \in [0,1)$ be defined by

$$\alpha = \frac{(m-j)(a-j/m)}{1-a+(m-j)(a-j/m)}, \tag{5.4.65}$$

so that by (5.4.61), (5.4.63) and (5.4.1)

$$\frac{1}{p} = \alpha\Big(\frac{1}{s} - \frac{1}{N}\Big) + \frac{1-\alpha}{t}.$$

It follows from Theorem 5.4.9 applied to $j^{th}$ order derivatives of $u$ that

$$|u|_{j,p} \le C|u|_{j+1,s}^{\alpha}|u|_{j,t}^{1-\alpha}. \tag{5.4.66}$$

We deduce from (5.4.66), (5.4.64) and (5.4.62) that

$$|u|_{j,p} \le C|u|_{m,r}^{\frac{\alpha+(j/m)(1-\alpha)(m-j)}{\alpha+(1-\alpha)(m-j)}} \|u\|_{L^q}^{\frac{(1-j/m)(1-\alpha)(m-j)}{\alpha+(1-\alpha)(m-j)}}.$$

Since by (5.4.65),

$$a = \frac{\alpha+(j/m)(1-\alpha)(m-j)}{\alpha+(1-\alpha)(m-j)},$$

this yields (5.4.2). □

Corollary 5.4.13 (Gagliardo-Nirenberg's inequality). *Let $\Omega \subset \mathbb{R}^N$ be an open subset. Let $1 \le p,q,r \le \infty$ and let $j,m$ be two integers, $0 \le j < m$. Assume that (5.4.1) holds for some $a \in [j/m,1]$ ($a<1$ if $r = N/(m-j) > 1$), and suppose further that $r < \infty$. It follows that $D^\alpha u \in L^p(\Omega)$ for all $u \in W_0^{m,r}(\Omega) \cap L^q(\Omega)$ if $|\alpha| = j$. Moreover, the inequality (5.4.2) holds for all $u \in W_0^{m,r}(\Omega) \cap L^q(\Omega)$.*

Proof. We first consider the case $\Omega = \mathbb{R}^N$. Let $u \in W^{m,r}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ and let $(u_n)_{n\ge 0} \subset C_c^\infty(\mathbb{R}^N)$ be the sequence constructed by regularization and truncation in the proof of Theorem 5.1.8, so that

$$u_n \underset{n\to\infty}{\longrightarrow} u \text{ in } W^{m,r}(\mathbb{R}^N) \quad \text{and} \quad \|u_n\|_{L^q} \le \|u\|_{L^q}. \tag{5.4.67}$$

Applying (5.4.2) to $u_n - u_\ell$, we obtain

$$|u_n - u_\ell|_{j,p} \le C|u_n - u_\ell|_{m,r}^a \|u_n - u_\ell\|_{L^q}^{1-a}. \tag{5.4.68}$$

Let $\alpha$ be a multi-index with $|\alpha| = j$. It follows from (5.4.67)-(5.4.68) that $D^\alpha u_n$ is a Cauchy sequence in $L^p(\mathbb{R}^N)$. Thus $D^\alpha u_n$ has a limit $v$ in $L^p(\mathbb{R}^N)$. In particular,

$$\int_{\mathbb{R}^N} D^\alpha u_n \varphi \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} v\varphi,$$

for all $\varphi \in C_c^j(\mathbb{R}^N)$. Since

$$\int_{\mathbb{R}^N} D^\alpha u_n \varphi = (-1)^j \int_{\mathbb{R}^N} u_n D^\alpha \varphi \underset{n\to\infty}{\longrightarrow} (-1)^j \int_{\mathbb{R}^N} u D^\alpha \varphi,$$

by (5.4.67), we see that $D^\alpha u = v \in L^p(\mathbb{R}^N)$ and

$$|u_n - u|_{j,p} \underset{n\to\infty}{\longrightarrow} 0, \tag{5.4.69}$$

which proves the first part of the result. Finally, we apply the inequality (5.4.2) to $u_n$. Letting $n \to \infty$ and using (5.4.67) and (5.4.69), we deduce that (5.4.2) holds for $u$. □

We are now in a position to state and prove the Sobolev embedding theorems. We restrict ourselves to functions of $W_0^{m,p}(\Omega)$. Similar statements hold for functions of $W^{m,p}(\Omega)$, but they are obtained by using extension operators, so they require a certain amount of regularity of the domain. For functions of $W_0^{m,p}(\Omega)$, instead, no regularity assumption on $\Omega$ is necessary. Furthermore, these results are sufficient for our purpose. Our first result in this direction is the following.

THEOREM 5.4.14. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, let $1 \le r < \infty$ and let $m \in \mathbb{N}$, $m \ge 1$.*

(i) *If $mr < N$, then $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ for all $p$ such that $r \le p \le Nr/(N - mr)$.*
(ii) *If $m = N$ and $r = 1$, then $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ for all $p$ such that $r \le p \le \infty$. Moreover, $W_0^{m,r}(\Omega) \hookrightarrow C_0(\Omega)$.*
(iii) *If $mr = N$ and $r > 1$, then $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ for all $p$ such that $r \le p < \infty$.*
(iv) *If $mr > N$, then $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ for all $p$ such that $r \le p \le \infty$. Moreover, $W_0^{m,r}(\Omega) \hookrightarrow C_0(\Omega)$.*

PROOF. The first embeddings of Properties (i)–(iv) follow from Corollary 5.4.13 by taking $j = 0$, $q = r$ and $a = N(p-r)/mpr$. The embeddings $W_0^{m,r}(\Omega) \hookrightarrow C_0(\Omega)$ in (ii) and (iv) follow from the density of $C_c^\infty(\Omega)$ in $W_0^{m,r}(\Omega)$ and the embedding $W_0^{m,r}(\Omega) \hookrightarrow L^\infty(\Omega)$. ☐

The next result is the general case of Sobolev's embedding for functions of $W_0^{m,p}(\Omega)$.

THEOREM 5.4.15. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, let $1 \le r < \infty$ and let $m, j \in \mathbb{N}$, $m \ge 1$.*

(i) *If $mr < N$, then $W_0^{m+j,r}(\Omega) \hookrightarrow W_0^{j,p}(\Omega)$ for all $p$ such that $r \le p \le Nr/(N - mr)$.*
(ii) *If $m = N$ and $r = 1$, then $W_0^{m+j,r}(\Omega) \hookrightarrow W_0^{j,p}(\Omega) \cap W^{j,\infty}(\Omega)$ for all $p$ such that $r \le p < \infty$. Moreover, $W_0^{m+j,r}(\Omega) \hookrightarrow C_0^j(\Omega)$.*
(iii) *If $mr = N$ and $r > 1$, then $W_0^{m+j,r}(\Omega) \hookrightarrow W_0^{j,p}(\Omega)$ for all $p$ such that $r \le p < \infty$.*
(iv) *If $mr > N$, then $W_0^{m+j,r}(\Omega) \hookrightarrow W_0^{j,p}(\Omega) \cap W^{j,\infty}(\Omega)$ for all $p$ such that $r \le p < \infty$. Moreover, $W_0^{m+j,r}(\Omega) \hookrightarrow C_0^j(\Omega)$.*

PROOF. We first prove (iv). Applying Theorem 5.4.14 (iv) to $D^\alpha u$ with $|\alpha| \le j$, we deduce that $W_0^{m+j,r}(\Omega) \hookrightarrow W^{j,p}(\Omega)$ for all $r \le p \le \infty$. The embedding $W_0^{m+j,r}(\Omega) \hookrightarrow W_0^{j,p}(\Omega)$ if $r \le p < \infty$ follows from the density of $C_c^\infty(\Omega)$ in $W_0^{m+j,r}(\Omega)$ and the embedding $W_0^{m+j,r}(\Omega) \hookrightarrow W^{j,p}(\Omega)$. Next, the embedding $W_0^{m+j,r}(\Omega) \hookrightarrow C_0^j(\Omega)$ follows from the density of $C_c^\infty(\Omega)$ in $W_0^{m+j,r}(\Omega)$ and the embedding $W_0^{m+j,r}(\Omega) \hookrightarrow W^{j,\infty}(\Omega)$. The proofs of (i), (ii) and (iii) are similar, using properties (i), (ii) and (iii) of Theorem 5.4.14, respectively. ☐

We now apply Morrey's inequality to obtain embedings in spaces of the type $C^{j,\alpha}(\overline{\Omega})$.

THEOREM 5.4.16. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, let $1 \le r < \infty$. Let $m \ge 1$ be the smallest integer such that $mr > N$. It follows that for all integers $j \ge 0$, $W_0^{m+j,r}(\Omega) \hookrightarrow C_0^j(\Omega) \cap C^{j,\alpha}(\overline{\Omega})$ with $\alpha = m - (N/r)$ if $(m-1)r < N$, $\alpha$ any number in $(0, 1)$ if $(m-1)r = N$.*

PROOF. Let $u \in C_c^\infty(\Omega)$. It follows from Theorem 5.4.15 (iv) that

$$\|u\|_{W^{j,\infty}} \le C\|u\|_{W^{m+j,r}}. \tag{5.4.70}$$

Let $\alpha$ be a multi-index with $|\alpha| = j$. Setting $v = D^\alpha u$, we see that

$$\|v\|_{W^{m,r}} \le \|u\|_{W^{m+j,r}}. \tag{5.4.71}$$

Let

$$\begin{cases} p = \frac{Nr}{N-(m-1)r} & \text{if } (m-1)r < N, \\ \max\{r, N\} < p < \infty & \text{if } (m-1)r = N, \end{cases}$$

so that $p > N$. It follows from Theorem 5.4.15 (i) and (5.4.71) that

$$\|v\|_{W^{1,p}} \le C\|v\|_{W^{m,r}} \le C\|u\|_{W^{m+j,r}}. \tag{5.4.72}$$

Finally, we deduce from (5.4.72) and Morrey's inequality (5.4.16) that

$$|v(x) - v(y)| \le C|x-y|^{1-\frac{N}{p}} \|u\|_{W^{m+j,r}},$$

for all $x, y \in \mathbb{R}^N$. Applying (5.4.70), we conclude that $\|u\|_{C^{j,\alpha}} \le C\|u\|_{W^{m+j,r}}$, with $\alpha = 1 - (N/p)$, which is the desired estimate. □

The following two results are applications of Sobolev's embedding theorems.

COROLLARY 5.4.17. *Given any* $1 \le r \le \infty$, $\bigcap\limits_{m \ge 0} W^{m,r}_{\mathrm{loc}}(\Omega) = C^\infty(\Omega)$.

PROOF. It is clear that $C^\infty(\Omega) \subset W^{m,r}_{\mathrm{loc}}(\Omega)$ for all $m \ge 0$. Conversely, suppose $u \in W^{m,r}_{\mathrm{loc}}(\Omega)$ for all $m \ge 0$. Let $\omega \subset\subset \Omega$ and let $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$ satisfy $\varphi(x) = 1$ for $x \in \omega$. It follows from Proposition 5.1.14 that $v = \varphi u \in W^{m,1}_0(\Omega)$ for all $m \ge 0$, so that $v \in C^\infty(\Omega)$ by Theorem 5.4.15. Thus $u \in C^\infty(\omega)$ and the result follows, since $\omega$ is arbitrary. □

PROPOSITION 5.4.18. *Let* $1 \le p \le \infty$, $m \in \mathbb{N}$, $m \ge 1$ *and* $u \in W^{m,p}_{\mathrm{loc}}(\Omega)$. *If* $D^\alpha u \in C(\Omega)$ *for all multi-index* $\alpha$ *with* $|\alpha| = m$, *then* $u \in C^m(\Omega)$.

PROOF. We proceed in three steps.

STEP 1. $u \in C(\Omega)$. Suppose $u \in L^{q_0}_{\mathrm{loc}}(\Omega)$ for some $q_0 \le N$ and let $q_0 \le q_1 < \infty$ satisfy

$$\frac{1}{q_1} \ge \frac{1}{q_0} - \frac{1}{N}.$$

Let $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$ and set $v = \varphi u$, so that $v \in L^{q_0}(\Omega)$. Since $\nabla v = \varphi \nabla u + u \nabla \varphi$ , we deduce that $\nabla v \in L^{q_0}(\Omega)$. $v$ being compactly supported in $\Omega$, it follows that $v \in W^{1,q_0}_0(\Omega)$ (see Remark 5.1.10 (i)). Applying Theorem 5.4.14, we see that $v \in L^{q_1}(\Omega)$ and, since $\varphi$ is arbitrary, we deduce that $u \in L^{q_1}_{\mathrm{loc}}(\Omega)$. We now iterate the above argument and, starting from $q_0 = 1$, we construct $q_0 < \cdots < q_{k-1} \le N < q_k$ such that $u \in L^{q_j}_{\mathrm{loc}}(\Omega)$ for $0 \le j \le k$. Finally, let $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$ and set $v = \varphi u$. We see as above that $v \in W^{1,q_k}(\Omega)$, and it follows from Theorem 5.4.14 that $v \in C(\Omega)$. Since $\varphi$ is arbitrary, we conclude that $u \in C(\Omega)$.

STEP 2. The case $m = 1$. It follows from Step 1 that $u \in C(\Omega)$. Since $\nabla u \in C(\Omega)$ by assumption, we have in particular $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Let $\omega \subset\subset \Omega$ and let $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$ satisfy $\varphi(x) = 1$ for $x \in \omega$. Set $v = \varphi u$, so that $v \in W^{1,\infty}(\Omega)$ by Proposition 5.1.14. Since $v$ is supported in a compact subset of $\Omega$, it follows that if

$$w(x) = \begin{cases} v(x) \text{ if } x \in \Omega, \\ 0 \text{ if } x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $w \in W^{1,\infty}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. Moreover, one verifies easily that

$$\nabla w = \begin{cases} u \nabla \varphi + \varphi \nabla u \text{ in } \Omega, \\ 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

so that $\nabla w \in C(\mathbb{R}^N)$. Since $w$ and $\nabla w$ have compact support, we see that $w, \nabla w \in C_{b,u}(\mathbb{R}^N)$. Applying Proposition 5.1.12, we deduce that $w \in C^1(\mathbb{R}^N)$, and since $w = u$ in $\omega$, it follows that $u \in C^1(\omega)$. Hence the result, since $\omega$ is arbitrary.

STEP 3. The case $m \ge 2$. We proceed by induction on $m$. By Step 2, the result holds for $m = 1$. Suppose it holds up to some $m \ge 1$. Let $u \in W^{m+1,p}_{\mathrm{loc}}(\Omega)$ satisfy $D^\alpha u \in C(\Omega)$ for all multi-index $\alpha$ with $|\alpha| = m + 1$. Consider $1 \le j \le N$ and set $v = \partial_j u$. It follows that $v \in W^{m,p}_{\mathrm{loc}}(\Omega)$ and $D^\alpha u \in C(\Omega)$ for all multi-index $\alpha$ with $|\alpha| = m$. Applying the result at the level $m$, we deduce that $v \in C^m(\Omega)$. Thus $\nabla u \in C^m(\Omega)$. In particular, $\nabla u \in C(\Omega)$ and we deduce from Step 2 that $u \in C^1(\Omega)$. Since $\nabla u \in C^m(\Omega)$, we conclude that $u \in C^{m+1}(\Omega)$. □

If $|\Omega| < \infty$, then $\|u\|_{L^r}$ is dominated only in terms of $\|\nabla u\|_{L^r}$ for functions of $W_0^{1,r}(\Omega)$. This is the object of the following result.

THEOREM 5.4.19 (Poincaré's inequality). *If $|\Omega| < \infty$ and $1 \le r < \infty$, then there exists a constant $C = C(N, r)$ (independent of $u$ and $\Omega$) such that*

$$\|u\|_{L^r} \le C|\Omega|^{\frac{1}{N}} \|\nabla u\|_{L^r}, \tag{5.4.73}$$

*for every $u \in W_0^{1,r}(\Omega)$.*

PROOF. Let $p = r(N+r)/N$, so that by (5.4.20), $\|u\|_{L^p} \le C\|\nabla u\|_{L^r}^{\frac{N}{N+r}} \|u\|_{L^r}^{\frac{r}{N+r}}$. Since $\|u\|_{L^r} \le |\Omega|^{\frac{1}{N+r}} \|u\|_{L^p}$ by Hölder's inequality, the result follows. □

COROLLARY 5.4.20. *Let $1 \le r < \infty$, and suppose $|\Omega| < \infty$. Then $\|u\| = \|\nabla u\|_{L^r}$ defines an equivalent norm on $W_0^{1,r}(\Omega)$.*

We end this section with a result concerning the embedding of $L^p$ spaces in negative order Sobolev spaces.

COROLLARY 5.4.21. *Suppose $\Omega \subset \mathbb{R}^N$ is an open set. Let $1 < r < \infty$ and let $m \ge 1$ be an integer. If*

$$\overline{p} = \begin{cases} \infty & \text{if} \quad mr \ge N, \\ \frac{Nr}{N-mr} & \text{if} \quad mr < N, \end{cases}$$

*then $L^{p'}(\Omega) \hookrightarrow W^{-m,r'}$ with dense embedding for all $r \le p \le \overline{p}$ (and $p < \infty$ if $mr = N$). If, in addition, $|\Omega| < \infty$, then the same property also holds for $1 \le p < r$.*

PROOF. The last part of the result (the case $|\Omega| < \infty$) follows from the dense embedding $L^p(\Omega) \hookrightarrow L^q(\Omega)$ if $1 \le p \le q < \infty$. The first part of the result follows from Theorem 5.4.14 and Proposition 5.1.19, except for the case $p = \infty$ (thus $mr > N$), since $L^1(\Omega)$ is not the dual of $L^\infty(\Omega)$. In this case, we argue directly as follows. It follows from Theorem 5.4.14 that $W_0^{m,r}(\Omega) \hookrightarrow L^\infty(\Omega)$. Define

$$eu(\varphi) = \int_\Omega u\varphi,$$

for all $u \in L^1(\Omega)$ and $\varphi \in W_0^{m,r}(\Omega)$. We have

$$|eu(\varphi)| \le \|u\|_{L^1} \|\varphi\|_{L^\infty} \le \|u\|_{L^1} \|\varphi\|_{W_0^{m,r}},$$

so that $e$ defines a mapping $L^1(\Omega) \to W^{-m,r'}(\Omega)$. This mapping is injective, because if $(eu, \varphi)_{W^{-m,r'}, W_0^{m,r}} = 0$ for all $\varphi \in W_0^{m,r}(\Omega)$, then in particular $\int_\Omega u\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, which implies $u = 0$. It remains to show that the embedding $e : L^1(\Omega) \to W^{-m,r'}(\Omega)$ is dense. To prove this, we observe that by the density of $C_c^\infty(\Omega)$ in $L^{r'}(\Omega)$ and of $L^{r'}(\Omega)$ in $W^{-m,r'}(\Omega)$ (see just above), it follows that $C_c^\infty(\Omega)$ is dense in in $W^{-m,r'}(\Omega)$. The result follows, since $C_c^\infty(\Omega) \subset L^1(\Omega)$. □

## 5.5. Compactness properties

We now study the compact embeddings of $W_0^{1,r}(\Omega)$. We begin with a local compactness result in $\mathbb{R}^N$.

PROPOSITION 5.5.1. *Let $1 \le r < \infty$ and let $K$ be a bounded subset of $W^{1,r}(\mathbb{R}^N)$. For every $R < \infty$, $K_R := \{u_{|B_R};\ u \in K\}$ is relatively compact in $L^r(B_R)$, where $B_R = B(0, R)$.*

PROOF. We proceed in three steps.

STEP 1. If $(\rho_n)_{n\geq 1}$ is a smoothing sequence, then

$$\|u-\rho_n\star u\|_{L^r}\leq \frac{C}{n}\|\nabla u\|_{L^r}, \tag{5.5.1}$$

for all $u\in W^{1,r}(\mathbb{R}^N)$, where $C=\left(\int_{\mathbb{R}^N}|y|^r\rho(y)\,dy\right)^{\frac{1}{r}}$. By density, we need only show (5.5.1) for $u\in C_c^\infty(\mathbb{R}^N)$. We claim that

$$\int_{\mathbb{R}^N}|u(x-y)-u(x)|^r\,dx\leq |y|^r\|\nabla u\|_{L^r}^r. \tag{5.5.2}$$

Indeed,

$$u(x-y)-u(x)=\int_0^1\frac{d}{dt}u(x-ty)\,dt=\int_0^1 y\cdot\nabla u(x-ty)\,dt;$$

and so,

$$|u(x-y)-u(x)|\leq |y|\int_0^1|\nabla u(x-ty)|\,dt\leq |y|\Bigl(\int_0^1|\nabla u(x-ty)|^r\,dt\Bigr)^{\frac{1}{r}}.$$

(5.5.2) follows after integration in $x$. Next, since $\|\rho_n\|_{L^1}=1$,

$$\begin{aligned}\rho_n\star u(x)-u(x)&=\int_{\mathbb{R}^N}\rho_n(y)(u(x-y)-u(x))\,dy\\&=\int_{\mathbb{R}^N}\rho_n(y)^{\frac{r-1}{r}}[\rho_n(y)^{\frac{1}{r}}(u(x-y)-u(x))]\,dy.\end{aligned}$$

By Hölder's inequality, we deduce

$$|\rho_n\star u(x)-u(x)|^r\leq\int_{\mathbb{R}^N}\rho_n(y)|u(x-y)-u(x)|^r\,dy.$$

Integrating the above inequality on $\mathbb{R}^N$ and applying (5.5.2), we find

$$\|\rho_n\star u-u\|_{L^r}^r\leq\|\nabla u\|_{L^r}^r\int_{\mathbb{R}^N}|y|^r\rho_n(y)\,dy.$$

Hence (5.5.1).

STEP 2. If $(\rho_n)_{n\geq1}$ is as in Step 1, then

$$\|\rho_n\star u\|_{W^{1,\infty}}\leq n^{\frac{N}{r}}\|\rho\|_{L^{r'}}\|u\|_{W^{1,r}}, \tag{5.5.3}$$

for all $u\in W^{1,r}(\mathbb{R}^N)$. Since $\nabla(\rho_n\star u)=\rho_n\star\nabla u$ by Lemma 5.1.9, it follows from Young's inequality that

$$\|\rho_n\star u\|_{W^{1,\infty}}\leq\|\rho_n\|_{L^{r'}}\|u\|_{W^{1,r}},$$

and the result follows.

STEP 3. Conclusion. Let $R>0$, let $K_R$ be as in the statement of the proposition, and let $\varepsilon>0$. Given $n\geq1$, set $K^n=\{\rho_n\star u;\,u\in K\}$ and $K_R^n=\{u_{|B_R};\,u\in K^n\}$. Fix $n$ large enough so that

$$\sup_{u\in K}\|u-\rho_n\star u\|_{L^r}\leq\frac{\varepsilon}{2}. \tag{5.5.4}$$

Such a $n$ exists by (5.5.1). It follows from (5.5.3) that $K^n$ is a set of uniformly Lipschitz continuous functions on $\mathbb{R}^N$. By Ascoli's theorem, $K_R^n$ is relatively compact in $L^\infty(B_R)$, thus in $L^r(B_R)$. Therefore, $K_R^n$ can be covered by a finite number of balls of radius $\varepsilon/2$ in $L^r(B_R)$. By (5.5.4), we see that $K_R$ can be covered by a finite number of balls of radius $\varepsilon$. Since $\varepsilon>0$ is arbitrary, this shows compactness. □

Corollary 5.5.2. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, let $1 \le r < \infty$ and let $(u_n)_{n\ge 0}$ be a bounded sequence of $W^{1,r}_0(\Omega)$. There exist $u \in L^r(\Omega)$ and a subsequence $(u_{n_k})_{k\ge 0}$ such that $u_{n_k} \to u$ a.e. in $\Omega$ and in $L^r(\Omega \cap B_R)$, for any $R < \infty$, as $k \to \infty$.*

Proof. We first consider the case $\Omega = \mathbb{R}^N$. It follows from Proposition 5.5.1 applied with $R = 1$ that there exist $n(1,k) \to \infty$ as $k \to \infty$ and $w_1 \in L^r(B_1)$ such that $u_{n(1,k)} \to w_1$ in $L^r(B_1)$ and a.e. in $B_1$. We now apply Proposition 5.5.1 with $R = 2$ to the sequence $(u_{n(1,k)})_{k\ge 0}$. It follows that there exist a subsequence $n(2,k) \to \infty$ as $k \to \infty$ and $w_2 \in L^r(B_2)$ such that $u_{n(2,k)} \to w_2$ in $L^r(B_2)$ and a.e. in $B_2$. By recurrence, we construct $n(\ell,k) \to \infty$ as $k \to \infty$ and $(w_\ell)_{\ell\ge 1}$ with $w_\ell \in L^r(B_\ell)$ such that $u_{n(\ell,k)} \to w_\ell$ in $L^r(B_\ell)$ and a.e. in $B_\ell$. Moreover, $(n(\ell,k))_{k\ge 0}$ is a subsequence of $(n(m,k))_{k\ge 0}$ for $\ell > m$, i.e. for every $k \ge 0$, there exists $k' \ge k$ such that $n(\ell,k) = n(m,k')$. We set $n_k = n(k,k)$. Since $n(k,k)$ is a subsequence of $n(\ell,k)$ for any $\ell \ge 1$, we see that $u_{n_k} \to w_\ell$ in $L^r(B_\ell)$ and a.e. in $B_\ell$. In particular, $w_\ell \equiv w_m$ on $B_m$ if $\ell \ge m$. We now set $u \equiv w_\ell$ on $B_m$, for $\ell \ge m$. We have $u_{n_k} \to u$ in $L^r(B_R)$ and a.e. in $B_R$, for any $R < \infty$. In particular,

$$\|u\|_{L^r(B_R)} = \lim_{k\to\infty} \|u_{n_k}\|_{L^r(B_R)} \le \limsup_{n\to\infty} \|u_n\|_{L^r(\mathbb{R}^N)}.$$

We deduce that $u \in L^r(\mathbb{R}^N)$. Since $u_{n_k} \to u$ a.e. in $B_R$ for any $R < \infty$, we conclude that $u_{n_k} \to u$ a.e. in $\mathbb{R}^N$.

We now consider the case of an arbitrary domain $\Omega \subset \mathbb{R}^N$. Let $(u_n)_{n\ge 0}$ be as above and set

$$\widetilde{u}_n(x) = \begin{cases} u_n(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

so that the sequence $(\widetilde{u}_n)_{n\ge 0}$ is bounded in $W^{1,r}(\mathbb{R}^N)$. (See Remark 5.1.10 (iv).) It follows from what precedes that there exist $\widetilde{u} \in L^r(\mathbb{R}^N)$, supported in $\Omega$, and a subsequence $(\widetilde{u}_{n_k})_{k\ge 0}$ such that $\widetilde{u}_{n_k} \to \widetilde{u}$ as $k \to \infty$ in $L^r(B_R)$ for any $R < \infty$ and a.e. in $\mathbb{R}^N$. The result now follows by setting $u = \widetilde{u}_{|\Omega}$. This completes the proof. □

Lemma 5.5.3. *Let $\Omega$ be an open subset of $\mathbb{R}^N$. Let $1 \le r \le \infty$, let $(u_n)_{n\ge 0}$ be a bounded sequence of $L^r(\Omega)$ and let $u \in L^1_{\mathrm{loc}}(\Omega)$. Suppose that*

$$\int_\Omega u_n \varphi \underset{n\to\infty}{\longrightarrow} \int_\Omega u\varphi, \tag{5.5.5}$$

*for all $\varphi \in C^\infty_c(\Omega)$ (which is satisfied in particular if $u_n \to u$ in $L^1(\omega)$ for every $\omega \subset\subset \Omega$). Then $u \in L^r(\Omega)$ and*

$$\|u\|_{L^r} \le \liminf_{n\to\infty} \|u_n\|_{L^r}. \tag{5.5.6}$$

*Moreover, if $r > 1$ then (5.5.5) holds for all $\varphi \in L^{r'}(\Omega)$. In addition, if $1 < r < \infty$ and if $\|u_n\|_{L^r} \to \|u\|_{L^r}$ as $n \to \infty$, then $u_n \to u$ in $L^r(\Omega)$.*

*Suppose further that $1 < r < \infty$ and that $(u_n)_{n\ge 0}$ is a bounded sequence of $W^{1,r}_0(\Omega)$. It follows that $u \in W^{1,r}_0(\Omega)$,*

$$\int_\Omega \nabla u_n \varphi \underset{n\to\infty}{\longrightarrow} \int_\Omega \nabla u\varphi, \tag{5.5.7}$$

*for all $\varphi \in L^{r'}(\Omega)$ and*

$$\|\nabla u\|_{L^r} \le \liminf_{n\to\infty} \|\nabla u_n\|_{L^r}. \tag{5.5.8}$$

*If in addition $\|\nabla u_n\|_{L^r} \to \|\nabla u\|_{L^r}$ as $n \to \infty$, then $\nabla u_n \to \nabla u$ in $L^r(\Omega)$.*

Proof. We claim that for all $u \in L^1_{\rm loc}(\Omega)$,

$$\|u\|_{L^r} = \sup\Bigl\{\Bigl|\int_\Omega u\varphi\Bigr|; \varphi \in C^\infty_{\rm c}(\Omega), \|\varphi\|_{L^{r'}} = 1\Bigr\}. \tag{5.5.9}$$

If $r > 1$, this is immediate because $L^{r'}(\Omega)^\star = L^r(\Omega)$ and $C^\infty_{\rm c}(\Omega)$ is dense in $L^{r'}(\Omega)$. Suppose now $r = 1$ and suppose $u \neq 0$ (the case $u = 0$ is immediate). Fix $0 < M < \|u\|_{L^1} \le \infty$. There exists a compact set $K \subset \Omega$ such that

$$\int_K |u| > M.$$

Let

$$h(x) = \begin{cases} \frac{u(x)}{|u(x)|} & \text{if} \quad u(x) \neq 0 \text{ and } x \in K, \\ 0 \quad \text{if} & u(x) = 0 \text{ or } x \notin K. \end{cases}$$

We have $h \in L^\infty(\Omega)$, $\|h\|_{L^\infty} = 1$. Moreover, $h$ has compact support in $\Omega$ and

$$\int_\Omega uh = \int_K |u| > M.$$

Let $(\rho_n)_{n\ge 0}$ be a smoothing sequence and set $h_n = (\rho_n \star h)_{|\Omega}$. For $n$ large enough, we have $h_n \in C^\infty_{\rm c}(\Omega)$. Moreover, up to a subsequence, $h_n \to h$ a.e. In addition, $\|h_n\|_{L^\infty} \le \|h\|_{L^\infty} = 1$. By dominated convergence, we deduce

$$\int_\Omega uh_n \underset{n\to\infty}{\longrightarrow} \int_\Omega uh > M;$$

and so,

$$\sup\Bigl\{\Bigl|\int_\Omega u\varphi\Bigr|; \varphi \in C^\infty_{\rm c}(\Omega), \|\varphi\|_{L^{p'}} = 1\Bigr\} \ge M.$$

Since $M < \|u\|_{L^1}$ is arbitrary, we deduce

$$\sup\Bigl\{\Bigl|\int_\Omega u\varphi\Bigr|; \varphi \in C^\infty_{\rm c}(\Omega), \|\varphi\|_{L^{p'}} = 1\Bigr\} \ge \|u\|_{L^1}.$$

The converse inequality being immediate, (5.5.9) follows. Now, since

$$\Bigl|\int_\Omega u_n\varphi\Bigr| \le \|u_n\|_{L^r}\|\varphi\|_{L^{r'}},$$

(5.5.6) follows from (5.5.5) and (5.5.9). The fact that if $r > 1$, then (5.5.5) holds for all $\varphi \in L^{r'}(\Omega)$ follows by density of $C^\infty_{\rm c}(\Omega)$ in $L^{r'}(\Omega)$.

Suppose now that $1 < r < \infty$, that $\|u_n\|_{L^r} \to \|u\|_{L^r}$ as $n \to \infty$ and let us show that $u_n \to u$ in $L^r(\Omega)$. If $\|u\|_{L^r} = 0$, then the result is immediate. Therefore, we may assume that $\|u\|_{L^r} \neq 0$, so that also $\|u_n\|_{L^r} \neq 0$ for $n$ large. Let then $\widetilde{u} = \|u\|^{-1}_{L^r} u$ and $\widetilde{u}_n = \|u_n\|^{-1}_{L^r} u_n$. It follows that

$$\|\widetilde{u}_n\|_{L^r} = \|\widetilde{u}\|_{L^r} = 1.$$

Furthermore, (5.5.5) is satisfied with $u$ and $u_n$ replaced by $\widetilde{u}$ and $\widetilde{u}_n$. Setting $w = 2\widetilde{u}$ and $w_n = \widetilde{u} + \widetilde{u}_n$, we deduce that (5.5.5) is satisfied with $u$ and $u_n$ replaced by $w$ and $w_n$. In particular, it follows from what precedes that $\|w\|_{L^r} \le \liminf \|w_n\|_{L^r}$. Since $\|w\|_{L^r} = 2$ and $\|w_n\|_{L^r} \le \|\widetilde{u}\|_{L^r} + \|\widetilde{u}_n\|_{L^r} = 2$, it follows that

$$\|w_n\|_{L^r} \underset{n\to\infty}{\longrightarrow} 2.$$

If $r \ge 2$, we have Clarkson's inequality (see e.g. Hewitt and Stromberg [26])

$$\|\widetilde{u}_n - \widetilde{u}\|^r_{L^r} \le 2^{r-1}(\|\widetilde{u}\|^r_{L^r} + \|\widetilde{u}_n\|^r_{L^r}) - \|\widetilde{u} + \widetilde{u}_n\|^r_{L^r}.$$

Therefore, $\|\widetilde{u}_n - \widetilde{u}\|_{L^r} \to 0$, from which it follows that $u_n \to u$ in $L^r(\Omega)$. In the case $r \leq 2$, the conclusion is the same by using Clarkson's inequality (see e.g. Hewitt and Stromberg [26])

$$\|\widetilde{u}_n - \widetilde{u}\|_{L^r}^{\frac{r}{r-1}} \leq 2(\|\widetilde{u}\|_{L^r}^r + \|\widetilde{u}_n\|_{L^r}^r)^{\frac{1}{r-1}} - \|\widetilde{u} + \widetilde{u}_n\|_{L^r}^{\frac{r}{r-1}}.$$

Suppose finally that $1 < r < \infty$ and that $(u_n)_{n\geq 0}$ is a bounded sequence of $W_0^{1,r}(\Omega)$. If $\varphi \in C_c^\infty(\Omega)$, then for all $j \in \{1, \dots, N\}$

$$-\int_\Omega \frac{\partial u_n}{\partial x_j}\varphi = \int_\Omega u_n \frac{\partial \varphi}{\partial x_j} \underset{n\to\infty}{\longrightarrow} \int_\Omega u\frac{\partial \varphi}{\partial x_j}, \tag{5.5.10}$$

by (5.5.5). Set now

$$f_j(\varphi) = -\int_\Omega u\frac{\partial \varphi}{\partial x_j},$$

for $\varphi \in C_c^\infty(\Omega)$. It follows from (5.5.10) and the boundedness of the sequence $(u_n)_{n\geq 0}$ in $W^{1,r}(\Omega)$ that

$$|f_j(\varphi)| \leq C\|\varphi\|_{L^{r'}}.$$

Therefore, $f$ can be extended by continuity and density to a linear, continuous functional on $L^{r'}(\Omega)$. Since $L^{r'}(\Omega)^\star = L^r(\Omega)$, there exists $g_j \in L^r(\Omega)$ such that

$$f_j(\varphi) = \int_\Omega g_j\varphi,$$

for all $\varphi \in C_c^\infty(\Omega)$ and by density, for all $\varphi \in L^{r'}(\Omega)$. This implies that

$$\int_\Omega u\frac{\partial \varphi}{\partial x_j} = -\int_\Omega g_j\varphi,$$

for all $\varphi \in C_c^\infty(\Omega)$. Thus $u \in W^{1,r}(\Omega)$. (5.5.7) follows from (5.5.10) and the above identity. The last properties follow by using (5.5.7) and applying the first part of the result to $\nabla u_n$ instead of $u_n$. □

We can now establish the compact sobolev embeddings.

THEOREM 5.5.4 (Rellich-Kondrachov). *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set, and let $1 \leq r < \infty$. It follows that the embedding $W_0^{1,r}(\Omega) \hookrightarrow L^r(\Omega)$ is compact.*

PROOF. Let $(u_n)_{n\geq 0}$ be a bounded sequence of $W_0^{1,r}(\Omega)$. It follows from Corollary 5.5.2 that there exist $u \in L^r(\Omega)$ and a subsequence $(u_{n_k})_{k\geq 0}$ such that $u_{n_k} \to u$ in $L^r(\Omega)$ as $k \to \infty$. This completes the proof. □

In fact, we have the following stronger result.

THEOREM 5.5.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set, let $1 \leq r < \infty$ and set*

$$\overline{r} = \begin{cases} \infty & \text{if} \quad r \geq N, \\ \frac{Nr}{N-r} & \text{if} \quad r < N. \end{cases}$$

*If $(u_n)_{n\geq 0}$ is a bounded sequence of $W_0^{1,r}(\Omega)$, then there is a subsequence $(u_{n_k})_{k\geq 0}$ and $u \in L^r(\Omega)$ such that $u_{n_k} \to u$ in $L^r(\Omega)$ as $k \to \infty$. Moreover, the following properties hold.*

(i) *$u \in L^{\overline{r}}(\Omega)$ ($u \in L^\rho(\Omega)$ for all $1 \leq \rho < \overline{r}$ if $r = N \geq 2$) and $u_n \to u$ in $L^p(\Omega)$ for all $1 \leq p < \overline{r}$.*
(ii) *If $r > 1$, then $u \in W_0^{1,r}(\Omega)$ and*

$$\int_\Omega \nabla u_{n_k}\varphi \underset{k\to\infty}{\longrightarrow} \int_\Omega \nabla u\varphi,$$

*for all* $\varphi \in L^{r'}(\Omega)$*. In particular,*

$$\|\nabla u\|_{L^r} \le \liminf_{k\to\infty} \|\nabla u_{n_k}\|_{L^r}.$$

*If, in addition,* $\|\nabla u\|_{L^r} = \lim \|\nabla u_{n_k}\|_{L^r}$ *as* $k \to \infty$*, then* $u_{n_k} \to u$ *in* $W_0^{1,r}(\Omega)$*.*

PROOF. The first part of the result follows from Theorem 5.5.4. Next, except in the case $r = N \ge 2$, it follows from Theorem 5.4.14 that $(u_n)_{n\ge 0}$ is bounded in $L^{\overline{r}}(\Omega)$, from which we deduce $u \in L^{\overline{r}}(\Omega)$. (See Lemma 5.5.3.) In the case $r = N \ge 2$, it follows from Theorem 5.4.14 that $(u_n)_{n\ge 0}$ is bounded in $L^p(\Omega)$ for any $p < \infty$, from which we deduce $u \in L^p(\Omega)$ for all $p < \infty$. Property (i) now follows from the $L^{\overline{r}}$ bound (or $L^p$ bound for all $p < \infty$ if $r = N \ge 2$) and the $L^r$ convergence by applying Hölder's inequality to $u_{n_k} - u$. Finally, property (ii) follows from Lemma 5.5.3. □

REMARK 5.5.6. If $\Omega$ is not bounded, we still have a local compactness result. Given $R > 0$, set $\Omega_R = \{x \in \Omega;\, |x < R|\}$. Given any bounded sequence $(u_n)_{n\ge 0}$ of $W_0^{1,r}(\Omega)$, there exist a subsequence $(u_{n_k})_{k\ge 0}$ and $u \in L^r(\Omega)$ such that $u_{n_k} \to u$ as $k \to \infty$, a.e. in $\Omega$ and in $L^r(\Omega_R)$ for every $R < \infty$. Moreover, the following properties hold.

(i) If $r = N = 1$, then $u \in L^\infty(\Omega)$ and $u_{n_k} \to u$ in $L^p(\Omega_R)$ for every $p < \infty$ and every $R < \infty$. In addition, $\|u\|_{L^p} \le \liminf \|u_n\|_{L^p}$ as $n \to \infty$ for every $1 \le p \le \infty$.

(ii) If $N \ge 2$ and $1 \le r < N$, then $u \in L^{\frac{Nr}{N-r}}(\Omega)$ and $u_{n_k} \to u$ in $L^p(\Omega_R)$ for every $p < Nr/(N-r)$ and every $R < \infty$. In addition, $\|u\|_{L^p} \le \liminf \|u_n\|_{L^p}$ as $n \to \infty$.

(iii) If $N \ge 2$ and $r = N$, then $u \in L^p(\Omega)$ and $u_{n_k} \to u$ in $L^p(\Omega_R)$ for every $p < \infty$ and every $R < \infty$. In addition, $\|u\|_{L^p} \le \liminf \|u_n\|_{L^p}$ as $n \to \infty$.

(iv) If $r > N$, then $u \in L^\infty(\Omega)$ and $u_{n_k} \to u$ in $L^\infty(\Omega_R)$ for every $R < \infty$. In addition, $\|u\|_{L^p} \le \liminf \|u_n\|_{L^p}$ as $n \to \infty$ for every $r \le p \le \infty$.

(v) If $r > 1$, then $u \in W_0^{1,r}\Omega)$ and

$$\int_\Omega \nabla u_{n_k} \varphi \underset{k\to\infty}{\longrightarrow} \int_\Omega \nabla u \varphi,$$

for all $\varphi \in L^{r'}(\Omega)$. In particular,

$$\|\nabla u\|_{L^r} \le \liminf_{k\to\infty} \|\nabla u_{n_k}\|_{L^r}.$$

If moreover $\|\nabla u\|_{L^r} = \lim \|\nabla u_{n_k}\|_{L^r}$ and $\|u\|_{L^r} = \lim \|u_{n_k}\|_{L^r}$ as $k \to \infty$, then $u_{n_k} \to u$ in $W_0^{1,r}(\Omega)$.

Those properties are proved like Theorem 5.5.5, except for the local convergence in (i)–(iv). This follows by applying Theorem 5.5.5 to the sequence $(\xi u_n)_{n\ge 0}$, where $\xi \in C_c^\infty(\mathbb{R}^N)$ is such that $\xi(x) = 1$ for $|x| \le R$.

COROLLARY 5.5.7. *Suppose* $\Omega \subset \mathbb{R}^N$ *is a bounded open set. Let* $1 < r < \infty$ *and let* $m \ge 1$ *be an integer. If*

$$\overline{p} = \begin{cases} \infty & \text{if} \quad mr \ge N, \\ \frac{Nr}{N-mr} & \text{if} \quad mr < N, \end{cases}$$

*then the embeddings* $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ *and* $L^{p'}(\Omega) \hookrightarrow W^{-m,r'}$ *are compact for all* $1 \le p < \overline{p}$*.*

PROOF. We first observe that by Theorem 5.5.4, the embedding $W_0^{1,r}(\Omega) \hookrightarrow L^r(\Omega)$ is compact, hence the embedding $W_0^{m,r}(\Omega) \hookrightarrow L^1(\Omega)$ is also compact. Applying Theorem 5.4.14 and Hölder's inequality, we deduce that if $1 \le p < \overline{p}$, then the embedding $W_0^{m,r}(\Omega) \hookrightarrow L^p(\Omega)$ is compact. This proves the first part of the result, and the second part follows from the abstract duality property of Proposition 5.1.19 (iii). ☐

## 5.6. Compactness properties in $\mathbb{R}^N$

In the case of unbounded domains, compactness may fail for various reasons. Consider for example the case $\Omega = \mathbb{R}^N$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varphi \not\equiv 0$, and let $y \in \mathbb{R}^N$, $y \neq 0$. Setting $u_n(x) = \varphi(x - ny)$, it is clear that $(u_n)_{n\geq 0}$ is bounded in $W^{1,r}(\mathbb{R}^N)$ for any $1 \leq r < \infty$. On the other hand, one sees easily that given any $R > 0$, $u_n \equiv 0$ on $B_R$ for $n$ large enough, so that the limit $u$ given by Corollary 5.5.2 is 0. On the other hand, $u_n \not\to u$ in $L^r(\mathbb{R}^N)$ since $\|u_n\|_{L^r} = \|\varphi\|_{L^r} \neq 0$. However, one sees that the sequence is relatively compact in $L^r(\mathbb{R}^N)$, up to a translation (since this is how the sequence was constructed).

One can also consider $z \in \mathbb{R}^N$, $z \neq 0$, $z \neq y$ and set $u_n(x) = \varphi(x - ny) + \varphi(x - nz)$. In this case, $u_n$ also converges locally to 0 and for $n$ large enough $\|u_n\|_{L^r} = 2\|\varphi\|_{L^r}$. However, in this case, one verifies easily that the sequence is not relatively compact up to a translation. Indeed the sequence splits in two parts, each of which is relatively compact up to translations, but with different translations.

Another case of noncompactness is the following. For $n \geq 1$, set $u_n(x) = n^{-\frac{N}{r}}\varphi(n^{-1}x)$. Then $\|u_n\|_{L^r} = \|\varphi\|_{L^r}$. On the other hand, $\|\nabla u_n\|_{L^r} = n^{-1}\|\nabla\varphi\|_{L^r}$, so that $u_n$ is bounded in $W^{1,r}(\mathbb{R}^N)$. However, $u_n$ converges locally to 0. In this case, the sequence is not relatively compact up to translations.

As a matter of fact, the three cases considered above describe the general situation, as follows from the following result. For simplicity, we consider the case $r = 2$, but a similar result holds in general. This result is based on the concentration compactness techniques introduced by P.-L. Lions [**32, 33**].

THEOREM 5.6.1. *Let $(u_n)_{n\geq 0}$ be a bounded sequence of $H^1(\mathbb{R}^N)$. Suppose there exists $a > 0$ such that $\|u_n\|_{L^2}^2 \to a$ as $n \to \infty$. It follows that there exists a subsequence $(u_{n_k})_{k\geq 0}$ which satisfies one of the following properties.*

(i) *(Compactness up to a translation.) There exist $u \in H^1(\mathbb{R}^N)$ and a sequence $(y_k)_{k\geq 0} \subset \mathbb{R}^N$ such that $u_{n_k}(\cdot - y_k) \to u$ in $L^p(\mathbb{R}^N)$ as $k \to \infty$, for $2 \leq p < 2N/(N-2)$ ($2 \leq p \leq \infty$ if $N = 1$).*

(ii) *(Vanishing.) $\|u_{n_k}\|_{L^p} \to 0$ as $k \to \infty$ for $2 < p < 2N/(N-2)$ ($2 < p \leq \infty$ if $N = 1$).*

(iii) *(Dichotomy.) There exist $0 < \mu < a$ and two sequences $(v_k)_{k\geq 0}$ and $(w_k)_{k\geq 0}$ of $H^1(\mathbb{R}^N)$ with compact support, such that*

$$\|v_k\|_{H^1} + \|w_k\|_{H^1} \leq C\sup\{\|u_n\|_{H^1};\, n \geq 0\},$$

*and*

$$\begin{aligned}
&\|v_k\|_{L^2}^2 \to \mu, \quad \|w_k\|_{L^2}^2 \to a - \mu,\\
&\operatorname{dist}(\operatorname{supp} v_k, \operatorname{supp} w_k) \to \infty,\\
&\|u_{n_k} - v_k - w_k\|_{L^p} \to 0 \text{ for } 2 \leq p < 2N/(N-2),\\
&\limsup \|\nabla v_k\|_{L^2}^2 + \|\nabla w_k\|_{L^2}^2 \leq \liminf \|\nabla u_{n_k}\|_{L^2}^2.
\end{aligned}$$

*as $k \to \infty$*

For the proof of Theorem 5.6.1, we will use the following estimate.

LEMMA 5.6.2. *If $2 < p < 2N/(N-2)$ ($2 < p \leq \infty$ if $N = 1$), then there exists a contant $C$ such that*

$$\|u\|_{L^p(\mathbb{R}^N)} \leq C\|u\|_{H^1(\mathbb{R}^N)}^\theta \Big(\sup_{y\in\mathbb{R}^N}\int_{|x-y|\leq 1}|u(x)|^2\,dx\Big)^{1-\theta},$$

*for every* $u \in H^1(\mathbb{R}^N)$. *Here,* $\theta = \max\{2/p, N(p-2)/2p\}$.

PROOF. Let $(Q_j)_{j\geq 1}$ be a sequence of disjoint unit cubes of $\mathbb{R}^N$ such that $\overline{\cup_{j\geq 0} Q_j} = \mathbb{R}^N$. Let $x_j$ be the center of the cube $Q_j$ and assume for example that $x_0 = 0$. Let $\rho \in C_c^\infty(\mathbb{R}^N)$ satisfy $\rho \equiv 1$ on $Q_0$ and $0 \leq \rho \leq 1$. Let $k$ be the number of cubes that intersect the support of $\rho$. Finally, set $\rho_j(x) = \rho(x - x_j)$ and $p_0 = 2 + 4/N$. It follows from Gagliardo-Nirenberg's inequality that

$$\|u\|_{L^{p_0}(Q_j)}^{p_0} \leq \|\rho_j u\|_{L^{p_0}(\mathbb{R}^N)}^{p_0} \leq C_1 \|\nabla(\rho_j u)\|_{L^2(\mathbb{R}^N)}^2 \|\rho_j u\|_{L^2(\mathbb{R}^N)}^{\frac{4}{N}}.$$

Summing in $j$, we obtain

$$\|u\|_{L^{p_0}(\mathbb{R}^N)} \leq C_1 \Big( \sum_{j\geq 0} \|\nabla(\rho_j u)\|_{L^2(\mathbb{R}^N)}^2 \Big) \sup_{j\geq 0} \|\rho_j u\|_{L^2(\mathbb{R}^N)}^{\frac{4}{N}}.$$

Since $\|\nabla(\rho_j u)\|_{L^2(\mathbb{R}^N)}^2 \leq C_2 \|u\|_{H^1(\operatorname{supp}(\rho_j))}^2$, we deduce

$$\sum_{j\geq 0} \|\nabla(\rho_j u)\|_{L^2(\mathbb{R}^N)}^2 \leq kC_2 \sum_{j\geq 0} \|u\|_{H^1(Q_j)}^2 \leq kC_2 \|u\|_{H^1(\mathbb{R}^N)}^2;$$

and so,

$$\|u\|_{L^{p_0}(\mathbb{R}^N)}^{p_0} \leq kC_1C_2 \|u\|_{H^1(\mathbb{R}^N)}^2 \sup_{j\geq 0} \|\rho_j u\|_{L^2(\mathbb{R}^N)}^{\frac{4}{N}}.$$

If $R$ is large enough so that $\operatorname{supp}\rho \subset B_R$, we deduce that

$$\|u\|_{L^{p_0}(\mathbb{R}^N)}^{p_0} \leq kC_1C_2 \|u\|_{H^1(\mathbb{R}^N)}^2 \Big( \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq R} |u(x)|^2\,dx \Big)^{\frac{4}{N}}.$$

Changing $u(x)$ to $u(R^{-1}x)$, we deduce

$$\|u\|_{L^{p_0}(\mathbb{R}^N)}^{p_0} \leq C \|u\|_{H^1(\mathbb{R}^N)}^2 \Big( \sup_{y\in\mathbb{R}^N} \int_{|x-y|\leq 1} |u(x)|^2\,dx \Big)^{\frac{4}{N}}.$$

The result now follows from Hölder's inequality between $\|u\|_{L^2}$ and $\|u\|_{L^{p_0}}$ if $2 < p \leq p_0$, and from Gagliardo-Nirenberg's inequality between $\|u\|_{L^{p_0}}$ and $\|\nabla u\|_{L^2}$ if $p > p_0$. ☐

PROOF OF THEOREM 5.6.1. Given $u \in H^1(\mathbb{R}^N)$, consider the distribution function of $u$,

$$\rho(t) = \sup_{y\in\mathbb{R}^N} \int_{\{|x-y|<t\}} |u(x)|^2\,dx, \tag{5.6.1}$$

for $t \geq 0$. It follows that $\rho$ is a nondecreasing function of $t$ and that $\rho(0) = 0$, $\rho(\infty) = \|u\|_{L^2}^2$. Moreover, we claim that for all $t \geq 0$, there exists $y(t)$ such that

$$\rho(t) = \int_{\{|x-y(t)|<t\}} |u(x)|^2\,dx. \tag{5.6.2}$$

Indeed, if $\rho(t) = 0$, we let $y(t) = 0$. If $\rho(t) > 0$, let $(y_j)_{j\geq 0}$ be a maximizing sequence in (5.6.1). We claim that $(y_j)_{j\geq 0}$ is bounded. Otherwise, we may extract a subsequence such that $\inf\limits_{0\leq k\leq j-1} \operatorname{dist}(y_j, y_k) \geq 2t$. It follows that

$$\int_{\mathbb{R}^N} |u(x)|^2\,dx \geq \sum_{j=0}^{\infty} \int_{\{|x-y_j|<t\}} |u(x)|^2\,dx = +\infty,$$

which is absurd. Therefore, there exists a subsequence $(y_{j_\ell})_{\ell\geq 0}$ which converges to $y(t)$, which clearly satisfies (5.6.2). Moreover, there exist $C < \infty$ and $\theta > 0$, independent of $u$, such that

$$|\rho(t) - \rho(s)| \leq C(t^{N-1} + s^{N-1})^\theta |t-s|^\theta \|u\|_{H^1}, \tag{5.6.3}$$

for all $s, t \geq 0$. Indeed, if $t > s$ then

$$\begin{aligned}\rho(t) - \rho(s) &= \int_{\{|x-y(t)|<t\}} |u(x)|^2\,dx - \int_{\{|x-y(s)|<s\}} |u(x)|^2\,dx \\ &\leq \int_{\{|x-y(t)|<t\}} |u(x)|^2\,dx - \int_{\{|x-y(t)|<s\}} |u(x)|^2\,dx \\ &= \int_{\{s<|x-y(t)|<t\}} |u(x)|^2\,dx.\end{aligned}$$

Given $q > 2$ such that $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, it follows that

$$\rho(t) - \rho(s) \leq |\{s < |x - y(t)| < t\}|^{\frac{q-2}{q}} \|u\|_{L^q}^2,$$

and (5.6.3) follows since $|\{s < |x - y(t)| \leq t\}| \leq C(t^{N-1} + s^{N-1})|t - s|$.

We now consider $(u_n)_{n\geq 0}$ as above, and we denote by $(\rho_n)_{n\geq 0}$ the corresponding distribution functions and by $y_n(t)$ the corresponding maximizers of (5.6.1). It follows from (5.6.3) that $(\rho_n)_{n\geq 0}$ is uniformly Hölder continuous on bounded intervals. By Ascoli's theorem, there exists a subsequence, which we still denote by $(\rho_n)_{n\geq 0}$, which converges to a function $\rho$ uniformly on bounded intervals. We have $\rho \geq 0$, $\rho(0) = 0$, and $\rho$ is nondecreasing. Let

$$\mu = \lim_{t\to\infty} \rho(t).$$

We clearly have

$$0 \leq \mu \leq a.$$

Furthermore, we claim that, by considering again a subsequence, there exists a sequence $(t_n)_{n\geq 0}$ such that $t_n > 0$, $t_n \to \infty$ and

$$\mu = \lim_{n\to\infty} \rho_n(t_n). \tag{5.6.4}$$

Indeed, for all $k \geq 1$, there exists $t_k \geq k$ such that $\rho(t_k) \geq \mu - 1/k$. Therefore, there exists $n_k \geq k$ such that $\mu - 2/k \leq \rho_{n_k}(t_k) \leq \mu + 1/k$. (5.6.4) follows by considering the subsequence $(\rho_{n_k})_{k\geq 0}$.

Next, we observe that for $n$ large enough, $\|u_n\|_{L^2} > \mu/2$, and we set

$$\widetilde{u}_n(x) = u_n(x - y_n(\overline{r})),$$

where $\overline{r}$ is such that $\rho(\overline{r}) > \mu/2$. In particular,

$$\rho_n(\overline{r}) \geq \mu/2,$$

for $n$ large enough. Applying Corollary 5.5.2 and Lemma 5.5.3 to the sequence $(\widetilde{u}_n)_{n\geq 0}$, we see that there exist a subsequence, which we denote again by $(\widetilde{u}_n)_{n\geq 0}$, and $u \in H^1(\mathbb{R}^N)$ such that $\widetilde{u}_n \to u$ in $L^2(B_R)$ for every $R > 0$. Moreover, $\|u\|_{L^2} \leq \liminf_{n\to\infty} \|\widetilde{u}_n\|_{L^2} = a$.

We now consider separately three cases.

CASE 1: $\mu = a$. We prove that in this case, (i) occurs. We first observe that

$$\|u\|_{L^2}^2 \leq a = \mu.$$

On the other hand, let $\mu/2 < \lambda < \mu$ and let $R$ be large enough so that $\rho(R) > \lambda$. It follows that for $n$ large enough, $\rho_n(R) \geq \lambda$. We claim that $|y_n(\overline{r}) - y_n(R)| \leq R + \overline{r}$. Indeed, otherwise the sets $\{|x - y_n(\overline{r})| < \overline{r}\}$ and $\{|x - y_n(R)| < R\}$ are disjoint, so that

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n(x)|^2\,dx &\geq \int_{\{|x-y_n(\overline{r})|<\overline{r}\}} |u_n|^2 + \int_{\{|x-y_n(R)|<R\}} |u_n|^2 \\ &= \rho_n(\overline{r}) + \rho_n(R) \geq \frac{\mu}{2} + \lambda > \mu,\end{aligned}$$

for $n$ large enough, which is absurd. Therefore, setting $\overline{R} = 2\overline{r} + 2R$, we have

$$\int_{\{|x|<\overline{R}\}} |\widetilde{u}_n(x)|^2\,dx \geq \int_{\{|x-y_n(R)|<R\}} |u_n(x)|^2\,dx = \rho_n(R) \geq \lambda,$$

for $n$ large. Since

$$\int_{\mathbb{R}^N} |u(x)|^2\,dx \geq \int_{\{|x|<\overline{R}\}} |u(x)|^2\,dx = \lim_{n\to\infty} \int_{\{|x|<\overline{R}\}} |\widetilde{u}_n(x)|^2\,dx,$$

we obtain $\|u\|_{L^2}^2 \geq \lambda$. Letting $\lambda \uparrow \mu$, we deduce $\|u\|_{L^2}^2 \geq \mu$, thus $\|u\|_{L^2}^2 = \mu$. By Lemma 5.5.3, this implies that $\widetilde{u}_n \to u$ in $L^2(\mathbb{R}^N)$, which proves (i) for $p = 2$, with $y_n = y_n(\overline{r})$. The case of an arbitrary $p$ as in (i) now follows from Gagliardo-Nirenberg's inequality.

Case 2: $\mu = 0$. We prove that in this case, (ii) occurs. Indeed, we have $\rho_n(1) \to 0$, so that

$$\sup_{y\in\mathbb{R}^N} \int_{\{|x-y|<1\}} |u_n(x)|^2\,dx \underset{n\to\infty}{\longrightarrow} 0.$$

Property (iii) now follows from Lemma 5.6.2.

Case 3: $0 < \mu < a$. We prove that in this case, (iii) occurs. We first show that

$$\mu = \lim_{n\to\infty} \rho_n(t_n/2). \tag{5.6.5}$$

Indeed, we have $\rho_n(t_n/2) \leq \rho_n(t_n)$, so that $\limsup \rho_n(t_n/2) \leq \mu$ by (5.6.4). On the other hand, let $t > 0$ and let $n$ be large enough so that $t_n/2 \geq t$. It follows that

$$\rho_n(t_n/2) \geq \rho_n(t) \underset{n\to\infty}{\longrightarrow} \rho(t);$$

and so, $\liminf \rho_n(t_n/2) \geq \rho(t)$. Letting $t \to \infty$, we deduce that $\liminf \rho_n(t_n/2) \geq \mu$, which proves (5.6.5). Next, we choose $\tau_n > 0$ such that

$$\int_{\{|x-y_n(t_n/2)|<\tau_n\}} |u_n|^2 \geq \|u_n\|_{L^2}^2 - \frac{1}{n}. \tag{5.6.6}$$

It follows that for $n$ large enough, $\tau_n > t_n$. Finally, let $\theta \in C^\infty([0,\infty))$ satisfy $\theta(t) \equiv 1$ for $0 \leq t \leq 1/2$, $\theta(t) \equiv 0$ for $t \geq 5/8$ and $0 \leq \theta \leq 1$. Let $\varphi \in C^\infty([0,\infty))$ be such that $\varphi(t) \equiv 0$ for $0 \leq t \leq 7/8$, $\varphi(t) \equiv 1$ for $t \geq 1$ and $0 \leq \varphi \leq 1$. For $n \geq 0$, let

$$\begin{cases} \theta_n(x) = \theta\Big(\frac{|x-y_n(t_n/2)|}{t_n}\Big), \\ \varphi_n(x) = \varphi\Big(\frac{|x-y_n(t_n/2)|}{t_n}\Big)\theta\Big(\frac{|x-y_n(t_n/2)|}{2\tau_n}\Big), \end{cases}$$

for $x \in \mathbb{R}^N$. Note that $\theta_n$ vanishes for $|x - y_n(t_n/2)| \geq 5t_n/8$ and that $\varphi_n$ vanishes for $|x - y_n(t_n/2)| \leq 7t_n/8$ and $|x - y_n(t_n/2)| \geq 5\tau_n/4$. Moreover, $\theta_n = 1$ for $|x - y_n(t_n/2)| \leq t_n/2$ and $\varphi_n = 1$ for $t_n \leq |x - y_n(t_n/2)| \leq \tau_n$ if $n$ is large enough so that $\tau_n > t_n$. In addition,

$$|\nabla\theta_n| + |\nabla\varphi_n| \leq C\Big(\frac{1}{t_n} + \frac{1}{\tau_n}\Big) \leq \frac{C}{t_n}. \tag{5.6.7}$$

We now define the sequences $(v_n)_{n\geq 0}$ and $(w_n)_{n\geq 0}$ by

$$v_n(x) = \theta_n(x)u_n(x),$$
$$w_n(x) = \varphi_n(x)u_n(x).$$

In particular, $(v_n)_{n\geq 0} \subset H^1(\mathbb{R}^N)$, $(w_n)_{n\geq 0} \subset H^1(\mathbb{R}^N)$, and both $v_n$ and $w_n$ have compact support. Moreover, one sees easily that

$$\operatorname{dist}(\operatorname{supp} v_n, \operatorname{supp} w_n) \geq t_n/8 \underset{n\to\infty}{\longrightarrow} +\infty, \tag{5.6.8}$$

and

$$\|v_n\|_{H^1} + \|w_n\|_{H^1} \leq C\|u_n\|_{H^1}. \tag{5.6.9}$$

Furthermore,

$$\rho_n(t_n/2) = \int_{\{|x-y_n(t_n/2)|<t_n/2\}} |u_n|^2 \le \int_{\mathbb{R}^N} |v_n|^2$$
$$\le \int_{\{|x-y_n(t_n/2)|<t_n\}} |u_n|^2 \le \int_{\{|x-y_n(t_n)|<t_n\}} |u_n|^2 = \rho_n(t_n),$$

so that

$$\|v_n\|_{L^2}^2 \underset{n\to\infty}{\longrightarrow} \mu, \tag{5.6.10}$$

by (5.6.4) and (5.6.5). Note also that $|u_n - v_n - w_n| \le |u_n|$. Since $u_n - v_n - w_n$ is supported in $\{t_n/2 < |x - y_n(t_n/2)| < t_n\} \cup \{|x - y_n(t_n/2)| > \tau_n\}$, we deduce, by applying (5.6.4), (5.6.5) and (5.6.6) that

$$\begin{aligned}\|u_n - v_n - w_n\|_{L^2}^2 &\le \int_{\{t_n/2<|x-y_n(t_n/2)|<t_n\}} u_n^2 + \int_{|x-y_n(t_n/2)|<\tau_n\}} u_n^2 \\ &\le \rho_n(t_n) - \rho_n(t_n/2) + \frac{1}{n},\end{aligned} \tag{5.6.11}$$

so that

$$\|u_n - v_n - w_n\|_{L^2}^2 \underset{n\to\infty}{\longrightarrow} 0. \tag{5.6.12}$$

By (5.6.10) and (5.6.8), we obtain in particular

$$\|w_n\|_{L^2}^2 \underset{n\to\infty}{\longrightarrow} a - \mu.$$

Next, by using Gagliardo-Nirenberg's inequality together with (5.6.9) and (5.6.12), we deduce that

$$\|u_n - v_n - w_n\|_{L^p} \underset{n\to\infty}{\longrightarrow} 0,$$

for $2 \le p < 2N/(N-2)$ ($2 \le p \le \infty$ if $N = 1$). Finally,

$$\begin{aligned}|\nabla u_n|^2 - |\nabla v_n|^2 - |\nabla w_n|^2 &= (1 - \theta_n^2 - \varphi_n^2)|\nabla u_n|^2 \\ &- (|\nabla\theta_n|^2 + |\nabla\varphi_n|^2)|u_n|^2 - 2(\theta_n\nabla\theta_n + \varphi_n\nabla\varphi_n)u_n\nabla u_n \\ &\ge -\frac{C}{t_n^2}|u_n|^2 - \frac{C}{t_n}|u_n|\,|\nabla u_n|;\end{aligned}$$

and so,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 - \int_{\mathbb{R}^N} |\nabla v_n|^2 - \int_{\mathbb{R}^N} |\nabla w_n|^2$$
$$\ge -\frac{C}{t_n^2}\|u_n\|_{L^2}^2 - \frac{C}{t_n}\|u_n\|_{L^2}\|\nabla u_n\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, (iii) is satisfied. □

For spherically symmetric functions in dimension $N \ge 2$, the situation is simpler, and we have the following compactness result of Strauss [**42**].

Theorem 5.6.3. *Suppose $N \ge 2$. If $(u_n)_{n\ge0} \subset H^1(\mathbb{R}^N)$ is a bounded sequence of spherically symmetric functions, then there exist a subsequence $(u_{n_k})_{k\ge0}$ and $u \in H^1(\mathbb{R}^N)$ such that $u_{n_k} \underset{k\to\infty}{\longrightarrow} u$ in $L^p(\mathbb{R}^N)$ for every $2 < p < 2N/(N-2)$ ($2 < p < \infty$ if $N = 2$).*

Proof. We proceed in three steps.

Step 1. Let $(u_n)_{n\ge0}$ be a bounded sequence of $H^1(\mathbb{R}^N)$. Assume that $u_n(x) \to 0$ as $|x| \to \infty$, uniformly in $n \ge 0$, i.e. for every $\varepsilon > 0$ there exists $R < \infty$ such that $|u_n(x)| \le \varepsilon$ for almost all $|x| \ge R$ and all $n \ge 0$. It follows that there exist a subsequence $(u_{n_k})_{k\ge0}$ and $u \in H^1(\mathbb{R}^N)$ such that $u_{n_k} \to u$ in $L^p(\mathbb{R}^N)$ as $k \to \infty$, for all $R < \infty$ and all $2 < p < 2N/(N-2)$.

Indeed, by Remark 5.5.6, there exist a subsequence $(u_{n_k})_{k\geq 0}$ and $u \in H^1(\mathbb{R}^N)$ such that $u_{n_k} \to u$ in $L^p(B_R)$ for any $R < \infty$ and a.e. in $\mathbb{R}^N$. In particular, $u(x)$ converges to 0 as $x \to \infty$. We have

$$\begin{aligned}\|u_{n_k} - u\|_{L^p(\mathbb{R}^N)} &= \|u_{n_k} - u\|_{L^p(B_R)} + \|u_{n_k} - u\|_{L^p(\{|x|>R\})} \\ &\leq \|u_{n_k} - u\|_{L^p(B_R)} + \|u_{n_k} - u\|_{L^\infty(\{|x|>R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Let $\delta > 0$. By uniform convergence, there exists $R < \infty$ such that

$$\|u_{n_k} - u\|_{L^\infty(\{|x|>R\})}^{\frac{p-2}{p}} \|u_{n_k} - u\|_{L^2(\mathbb{R}^N)} \leq \frac{\delta}{2}.$$

Next, $R$ being chosen, for $k$ large enough we have

$$\|u_{n_k} - u\|_{L^p(B_R)} \leq \frac{\delta}{2};$$

and so, $\|u_{n_k} - u\|_{L^p(\mathbb{R}^N)} \leq \delta$. Hence the result.

Step 2. If $u \in H^1(\mathbb{R}^N)$ is spherically symmetric, then

$$|x|^{\frac{N-1}{2}} |u(x)| \leq \sqrt{2} \|u\|_{L^2} \|\nabla u\|_{L^2}, \tag{5.6.13}$$

for a.a. $x \in \mathbb{R}^N$.

By truncation and regularisation, there exists a sequence of spherically symmetric functions $(u_n)_{n\geq 0} \subset C_c^\infty(\mathbb{R}^N)$ such that $u_n \to u$ as $n \to \infty$, in $H^1(\mathbb{R}^N)$ and a.e. Therefore, we need only establish the estimate for spharically symmetric, smooth functions. In this case,

$$r^{N-1} u(r)^2 = -\int_r^\infty \frac{d}{ds}(s^{N-1} u(s)^2)\, ds \leq 2 \int_r^\infty s^{N-1} u(s) u'(s)\, ds.$$

The result now follows from Cauchy-Schwarz inequality.

Step 3. Conclusion. We deduce from the estimate (5.6.13) that $u_n(x) \to 0$ as $|x| \to \infty$, uniformly in $n \geq 0$, and the conclusion follows from Step 1. □

Remark 5.6.4. The conclusion of Theorem 5.6.3 could also be obtained by using Theorem 5.6.1 and the estimate (5.6.13).

Remark 5.6.5. Suppose $N \geq 2$. Let $R > 0$ and $\Omega = \{x \in \mathbb{R}^N;\, |x| > R\}$. Let $W$ be the subspace of $H_0^1(\Omega)$ of radially symmetric functions. Given $u \in W$, we may extend $u$ by 0 outside $\Omega$ in order to obtain a radially symmetric function of $H^1(\mathbb{R}^N)$. By applying (5.6.13), we deduce that $W \hookrightarrow L^\infty(\Omega)$. Arguing as in Theorem 5.6.3, one shows that if $(u_n)_{n\geq 0} \subset W$ is a bounded sequence, then there exist a subsequence $(u_{n_k})_{k\geq 0}$ and $u \in W$ such that $u_{n_k} \to u$ in $L^p(\mathbb{R}^N)$ as $k \to \infty$, for every $2 < p < \infty$.

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