Timoshenko systems with indefinite damping✩

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Abstract
We consider the Timoshenko system in a bounded domain \((0, L) \subset \mathbb{R}^1\). The system has an indefinite damping mechanism, i.e. with a damping function \(a = a(x)\) possibly changing sign, present only in the equation for the rotation angle. We shall prove that the system is still exponentially stable under the same conditions as in the positive constant damping case, and provided \(\bar{a} = \int_0^L a(x) \, dx > 0\) and \(\|a - \bar{a}\|_2 < \epsilon\), for \(\epsilon\) small enough. The decay rate will be described explicitly. In the arguments, we shall also give a new proof of exponential stability for the constant case \(a \equiv \bar{a}\). Moreover, we give a precise description of the decay rate and demonstrate that the system has the spectrum determined growth (SDG) property, i.e. the type of the induced semigroup coincides with the spectral bound for its generator.

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1. Introduction
Here we will consider the system
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0 \quad \text{in } (0, \infty) \times (0, L), \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + a(x)\psi_t &= 0 \quad \text{in } (0, \infty) \times (0, L),
\end{align*}
with positive constants \(\rho_1, k, \rho_2, b, \gamma, \rho_3, \kappa\) together with initial conditions
\begin{align*}
\varphi(0, \cdot) &= \varphi_0, \quad \varphi_t(0, \cdot) = \varphi_1, \quad \psi(0, \cdot) = \psi_0, \quad \psi_t(0, \cdot) = \psi_1, \quad \theta(0, \cdot) = \theta_0 \quad \text{in } (0, L), \tag{1.3}
\end{align*}
and boundary conditions
\begin{align*}
\varphi(t, 0) = \varphi(t, L) = \psi_x(t, 0) = \psi_x(t, L) = 0 \quad \text{in } (0, \infty). \tag{1.4}
\end{align*}

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It models the transverse displacement $\varphi$ of a beam with reference configuration $(0, L) \subset \mathbb{R}^1$ and the rotation angle $\psi$ of a filament. The well-posedness of (1.1)–(1.4) is standard, cf. [17].

There is a damping mechanism present (only) in one equation, (1.2), given by $a(x)\psi_t$, where $a \in L^\infty((0, L))$ may change sign, but will satisfy

$$\bar{a} := \frac{1}{L} \int_0^L a(x) \, dx > 0. \tag{1.5}$$

For strictly positive $a$ it was shown by Soufyane [17] that the system is exponentially stable if and only if

$$\frac{\rho_1}{k} = \frac{\rho_2}{b} \tag{1.6}$$

holds. That is under this condition the damping in only one equation is strong enough for the exponential decay of the associated energy

$$E(t) := \frac{1}{2} \int_0^L \left( \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + b \psi_x^2 + k |\varphi_x + \psi|^2 \right)(t, x) \, dx$$

$\equiv E(t, \varphi, \psi)$

(mostly dropping $(t, x)$ in the sequel). In the author’s paper [1] the damping $a\psi_t$ could be replaced by a memory term $\int_0^t g(t-s)\psi_{xx} \, ds$, and in [12] by a coupling to a heat equation, see also [13] for nonlinear systems.

Already for the wave equation

$$u_{tt} - u_{xx} + a(x)u_t = 0,$$

with Dirichlet boundary conditions, it is a subtle issue to see whether an indefinite damping with the function $a$ just satisfying (1.5) still leads to exponential stability, and which additional conditions have to be added, respectively. The nondissipative case with indefinite $a$ seems to have been posed first by Chen, Fulling, Narcovich and Sun [3] where it was conjectured that the energy

$$E_0(t) = \int_0^L (u_t^2 + u_x^2)(t, x) \, dx$$

decays exponentially if

$$\exists \gamma > 0, \forall n = 1, 2, \ldots : \int_0^L a(x) \sin^2(n\pi x/L) \, dx \geq \gamma$$

holds. Later Freitas [4] found that the latter condition on the moments is not sufficient to guarantee exponential stability when $\|a\|_{L^\infty}$ is large, but replacing $a$ by $\varepsilon a$, Freitas and Zuazua [6] proved that when $a$ is of bounded variation and the condition on the moments holds, then there is $\varepsilon^* = \varepsilon^*(a)$ such that for all $\varepsilon \in (0, \varepsilon^*)$ the energy decays indeed exponentially. This result was extended to a differential equation of the type

$$u_{tt} - u_{xx} + \varepsilon a(x)u_t + b(x)u = 0 \tag{1.7}$$

by Benaddi and Rao [2]. K. Liu, Z. Liu and Rao [8] gave an abstract treatment of these results under certain conditions on the abstract damping operator. An extension to higher space dimensions was presented by Liu, Rao and Zhang [9], for unbounded domains see the recent work of Freitas and Krejčiřík [5].

In [14] the authors could show that is sufficient to require just

$$\|a - \bar{a}\|_{L^2} \quad \text{small enough.} \tag{1.8}$$

---

1 In [10] it was pointed out that this conditions for real materials never holds, but the analysis gives insight for various problems.
There it was also shown that there are certain pairs \((a, L)\) with possibly negative moments 
\[ \int_0^L a(x) \sin^2(n\pi x/L) \, dx \]
but still leading to exponential decay. An extension to the type of Eq. (1.7) was given by Menz [11].

For the Timoshenko system under consideration, we shall demonstrate that the conditions (1.5)–(1.8) are sufficient to yield exponential stability.

Moreover, we shall precisely describe the best rate of decay \(d_0\) for the energy in the estimate

\[ \exists d_0 > 0, \, \exists C_0 > 0, \, \forall t \geq 0: \quad E(t) \leq C_0 E(0) e^{-2d_0 t} \]  

for the constant coefficient case \(a = \bar{a}\). This is related to the result to be proved here that the system has the so-called spectrum determined growth property (SDG property); that is, after having reformulated the system as a first-order system (see Section 2) \(V_t = AV\) with a \(C_0\)-semigroup generator \(A\) in an appropriate Hilbert space, we shall prove that the type of the semigroup, the growth abscissa \(\omega_0(A)\), equals the spectral bound \(\omega_{\sigma}(A)\),

\[ \omega_0(A) = \omega_{\sigma}(A). \]  

Here, in general, the type of a \(C_0\)-semigroup generator \(G\) is defined as

\[ \omega_0(G) = \lim_{t \to \infty} \frac{\ln \| e^{Gt} \|}{t} = \inf_{t > 0} \frac{\ln \| e^{Gt} \|}{t}, \]  

and the spectral bound is given as the least upper bound for the real parts of the values in the spectrum \(\sigma(G)\) of \(G\),

\[ \omega_{\sigma}(G) = \sup \{ \Re \lambda \mid \lambda \in \sigma(G) \}. \]  

For a discussion of the SDG property see the work of Prüss [15], Huang [7] and Renardy [16]. We shall prove the SDG property for our system using a characterization given by Prüss [15] or Huang [7], saying

\[ \omega_0(G) = \inf \{ \omega \in \mathbb{R} \mid \exists M = M(\omega), \, \forall \lambda, \, \Re \lambda \geq \omega: \| (\lambda - G)^{-1} \| \leq M \}. \]  

To prove our results for \(A\), we shall regard it as a perturbation of \(\bar{A}\), where \(\bar{A}\), as above, represents the same system but with \(a(x)\) being replaced by a (the) constant \(\bar{a} > 0\). This corresponds to a previously discussed constant damping in one equation, cf. [13,17]. Here it will be demonstrated that it yields exponential stability—first another proof of this known result using (1.13), but second and additionally showing that

\[ \omega_0(A) = \omega_{\sigma}(A) \]  

holds, and in determining \(\omega_{\sigma}(A)\) explicitly. An explicit representation of the inverse \((\lambda - A)^{-1}\) and a sophisticated analysis of \(\| (\lambda - A)^{-1} \|\) will yield the result. Then a fixed point argument for \(A\), where (1.8) will describe the contraction precisely, will be used.

We remark that a stronger requirement of the smallness of \(\| a \|_{L^\infty}\) as discussed for wave equations by other authors, see [2,6,9] could be treated in an even easier way for our system too.

We also remark that our technique does not immediately carry over to the case where also \(\psi\) satisfies Dirichlet boundary conditions, although similar results can be expected.

Summarizing our new contributions are assuming (1.6),

- to show that in the positive constant damping case, the system has the SDG property, and to give for this situation a precise computation of the rate of decay (and the type of the semigroup),
- to show that also for the case of possibly indefinite damping, the system is still exponentially stable, for small \(\| a - \bar{a} \|_{L^2}\).

The paper is organized as follows. In Section 2 we shall formulate the semigroup setting, in Section 3 we discuss the constant coefficient case yielding the SDG property there, and in Section 4 we finish the discussion of the original indefinite problem obtaining the result on exponential stability.
2. The semigroup setting

We rewrite the initial–boundary value problem (1.1)–(1.4) as a first-order system for $V := (\varphi, \varphi_t, \psi, \psi_t)'$, where the prime is used to denote the transpose. Then $V$ satisfies

$$V_t = AV, \quad V(t = 0) = V_0,$$

where $V_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1)'$ and $A$ is the (formal) differential operator

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
k/\rho_1 \partial_x^2 & 0 & k/\rho_1 \partial_x & 0 \\
0 & 0 & 0 & 1 \\
-k/\rho_2 \partial_x & 0 & b/\rho_2 \partial_x^2 - k/\rho_2 & -a/\rho_2
\end{pmatrix}.$$  \hspace{1cm} (2.2)

Let

$$H := H^1_0((0, L)) \times L^2((0, L)) \times H^1_0((0, L)) \times L^2((0, L))$$

be the Hilbert space with norm given by

$$\|V\|_H^2 = \|(\varphi^1, \varphi^2, \psi^1, \psi^2)'\|_H^2$$

$$\equiv \rho_1 \|\varphi^2\|_{L^2}^2 + b \|\psi^1\|_{L^2}^2 + k \|\varphi^1 + \psi^1\|_{L^2}^2 + \rho_2 \|\psi^2\|_{L^2}^2.$$  \hspace{1cm} (2.3)

Then $A$, formally given in (2.2), with domain

$$D(A) := \{ V \in H \mid \varphi^1 \in H^2((0, L)), \quad \varphi^2 \in H^1_0((0, L)), \quad \psi^1 \in H^2((0, L)), \quad \psi^2 \in H^1_0((0, L)), \quad \psi^1_{xx} \in L^2((0, L)) \},$$

generates a semigroup $\{e^{tA}\}_{t \geq 0}$. We observe that for a solution $(\varphi, \psi)$ to (1.2)–(1.4), and the corresponding $V$, the norm $\|V(t)\|_H^2$ equals twice the energy $E(t)$ of $(\varphi, \psi)$ defined by

$$E(t) := \frac{1}{2} \int_0^L \left( \rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + b |\psi_x|^2 + k |\varphi_x + \psi|^2 \right)(t, x) \, dx.$$  \hspace{1cm} (2.3)

Replacing the function $a = a(x)$ in (1.2) by the constant $\bar{a}$, we write $A$ for the arising constant coefficient operator instead of $A$. We shall first give in the next section a precise description of the spectrum of $A$, and we show that the SDG property holds for $A$.

3. The constant coefficient case

Since it is not difficult to see that the inverse of the operator $A$ is compact, we have to determine the eigenvalues of $A$ in a way that allows us to determine $\omega_r(A)$ and to estimate the resolvent operators uniformly. Therefore, let

$$(A - \lambda)W = 0$$

with $\lambda \in \mathbb{C} \setminus \{0\}$ and $W \in D(A)$. Then $W = (\varphi, \lambda \varphi, \psi, \lambda \psi)'$, and $(\varphi, \psi)$ satisfy

$$\rho_1 \lambda^2 \varphi - k(\varphi_x + \psi)_x = 0 \quad \text{in } (0, L), \hspace{1cm} (3.1)$$

$$\rho_2 \lambda^2 \psi - b\psi_{xx} + k(\varphi_x + \psi) + \bar{a} \lambda \psi = 0 \quad \text{in } (0, L), \hspace{1cm} (3.2)$$

together with the boundary conditions

$$\varphi(0) = \varphi(L) = \psi_x(0) = \psi_x(L) = 0.$$  \hspace{1cm} (3.3)

Observing the boundary conditions, we can reduce the system for $(\varphi, \psi)$ to a single one for $\varphi$ by differentiating (3.1) and using (3.2), yielding
\[ bk\varphi_{xxx} - \left[(k\rho_2 + b\rho_1)\lambda^2 + k\bar{a}\lambda\right]\varphi_{xx} + \left(\rho_2\lambda^2 + \bar{a}\lambda + k\right)\rho_1\lambda^2\varphi = 0, \quad (3.4) \]
\[
\varphi(0) = \varphi(L) = \varphi_{xx}(0) = \varphi_{xx}(L) = 0. \quad (3.5) 
\]

For (3.4), (3.5), we have a complete orthonormal system of eigenfunctions: \( \varphi_j(x) = \sqrt{\frac{2}{L}} \sin(\theta_j x) \) with \( \theta_j := \frac{j\pi}{L} \).

For (3.4), (3.5), we have a complete orthonormal system of eigenfunctions: \( \varphi_j(x) = \sqrt{\frac{2}{L}} \sin(\theta_j x) \) with \( \theta_j := \frac{j\pi}{L} \).

Then \( \lambda = \lambda_j \) has to satisfy
\[
P(\lambda, \theta_j) \equiv \rho_1\rho_2\lambda^4 + \bar{a}\rho_1\lambda^3 + \left[(k\rho_2 + b\rho_1)\theta_j^2 + k\rho_1\right]\lambda^2 + k\bar{a}\theta_j^2\lambda + bk\theta_j^4 = 0. 
\]

Dividing by \( \rho_1\rho_2\lambda^2 \) we find
\[
\lambda^2 + \frac{k}{\rho_1\rho_2} \theta_j^4 \lambda^2 + \left[\left(\frac{k}{\rho_1} + \frac{b}{\rho_2}\right)\theta_j^2 + \frac{k}{\rho_2}\right] + \frac{\bar{a}}{\rho_1\rho_2} \theta_j^2 \lambda + \frac{bk}{\rho_1\rho_2} \theta_j^4 = 0. 
\]

Rearranging terms we get
\[
\lambda^2 + \frac{bk}{\rho_1\rho_2} \theta_j^4 \frac{1}{\lambda^2} + \left[\left(\frac{k}{\rho_1} + \frac{b}{\rho_2}\right)\theta_j^2 + \frac{k}{\rho_2}\right] + \frac{\bar{a}}{\rho_1\rho_2} \left\{\frac{k\theta_j^2}{\rho_1} \frac{1}{\lambda} + \lambda\right\} = 0. 
\]

Now assuming and using the identity (1.6), and defining
\[
y := \frac{k\theta_j^2}{\rho_1} \frac{1}{\lambda} + \lambda, \quad (3.7) 
\]

we obtain from (3.6)
\[
y^2 + \frac{\bar{a}}{\rho_2} y + \frac{k}{\rho_2} = 0. \quad (3.8) 
\]

This implies
\[
y \equiv y_{1,2} = -\frac{\bar{a}}{2\rho_2} \pm \sqrt{\frac{\bar{a}^2}{4\rho_2^2} - \frac{k}{\rho_2}}. \quad (3.9) 
\]

Multiplying (3.7) by \( \lambda \) we get
\[
\lambda^2 - \lambda y + \frac{k\theta_j^2}{\rho_1} = 0, \quad (3.10) 
\]

implying
\[
\lambda = \frac{y}{2} \pm \sqrt{\frac{y^2}{4} - \frac{k\theta_j^2}{\rho_1}}. \quad (3.11) 
\]

Therefore the following set:
\[
B := \left\{\frac{y}{2} \pm \sqrt{\frac{y^2}{4} - \frac{k\theta_j^2}{\rho_1}} \bigg| y = -\frac{\bar{a}}{2\rho_2} \pm \sqrt{\frac{\bar{a}^2}{4\rho_2^2} - \frac{k}{\rho_2}}, \ \theta_j = \frac{j\pi}{L}, \ \ j \in \mathbb{N}\right\} \quad (3.12) 
\]

is the candidate for the (point) spectrum of \( A \). For \( \lambda_j \in B \), a possible eigenfunction \( W \) has the form \( W_j = \text{const.} \left(\varphi_j, \lambda_j \varphi_j, \psi_j, \lambda_j \psi_j\right)' \), where \( \psi_j \) is determined from (3.1), (3.2) as
\[
\psi_j = \psi_j(x) = \text{const.} \frac{b\rho_1\theta_j\lambda_j^2 + bk\theta_j^3 - k\theta_j}{k(\rho_2\lambda_j^2 + k + \bar{a}\lambda_j)} \cos(\theta_j x), 
\]

provided
\[
\rho_2\lambda_j^2 + k + \bar{a}\lambda_j \neq 0. \quad (3.13) 
\]

The validity of (3.13) can be seen as follows.
Observing (3.8), (3.13) and
\[ \lambda_j = \lambda_r^j, \quad r = 1, 2, 3, 4, \]
with
\[ \lambda_{1,2}^j = \frac{\gamma_1}{2} \pm \sqrt{\frac{\gamma_1^2}{4} - \frac{k\theta_j^2}{\rho_1}}, \quad \lambda_{3,4}^j = \frac{\gamma_2}{2} \pm \sqrt{\frac{\gamma_2^2}{4} - \frac{k\theta_j^2}{\rho_1}} \]
(3.14)
we have to show
\[ \lambda_r^j \neq y_m \quad \text{for} \quad r = 1, 2, 3, 4 \quad \text{and} \quad m = 1, 2. \]
Let w.l.o.g. \( m = 1 \). Then, by (3.8), (3.13) we immediately have \( \lambda_1^j \neq y_1, \lambda_2^j \neq y_1 \). The assumption \( \lambda_3^j = y_1 \) is equivalent to
\[ \frac{\gamma_2}{2} + \sqrt{\frac{\gamma_2^2}{4} - \frac{k\theta_j^2}{\rho_1}} = y_1. \]
(3.15)
We can exclude the case that \( \bar{a}^2 = 4\rho_2 k \), because this would imply that \( \lambda_1^j = \lambda_3^j = y_1 \), a contradiction. For the case that \( \bar{a}^2 > 4\rho_2 k \) we conclude
\[ 0 > y_1 > y_2 > \frac{\gamma_2}{2} + \sqrt{\frac{\gamma_2^2}{4} - \frac{k\theta_j^2}{\rho_1}} = y_1 \]
again a contradiction (\( \sqrt{\ldots} \) being imaginary is impossible too).

In the last case that \( \bar{a}^2 < 4\rho_2 k \), we conclude
\[ \sqrt{\frac{\gamma_2^2}{4} - \frac{k\theta_j^2}{\rho_1}} = y_1 - \frac{\gamma_2}{2} = -\bar{a} + i\sqrt{4\rho_2 k - \bar{a}^2} \]
implies, after taking squares and comparing the imaginary part,
\[ 8\bar{a} \sqrt{4\rho_2 k - \bar{a}^2} = 0, \]
or \( 4\rho_2 k = \bar{a}^2 \), again a contradiction.

Altogether we proved (3.13) and hence

**Theorem 3.1. Assume (1.6). Then**

\[ \sigma(A) = \left\{ \frac{y}{2} \pm \sqrt{\frac{\gamma_2^2}{4} - \frac{k\theta_j^2}{\rho_1}} \mid y = -\frac{\bar{a}}{2\rho_2} \pm \sqrt{\frac{\bar{a}^2}{4\rho_2^2} - \frac{k}{\rho_2}}, \quad \theta_j = j\pi, \quad j \in \mathbb{N} \right\}. \]

Next we determine \( \omega_{\sigma}(A) \) explicitly.

**Case 1. \( \bar{a}^2 \geq 4\rho_2 k \).**

This implies
\[ y_{1/2} \in \mathbb{R}, \quad 0 > y_1 \geq y_2, \]
hence
\[ \max_{j \in \mathbb{N}} \max_{r=1,2,3,4} \text{Re} \lambda_r^j = \text{Re} \left\{ \frac{y_1}{2} \pm \sqrt{\frac{\gamma_1^2}{4} - \frac{k\theta_j^2}{\rho_1}} \right\} \]
\[ = -\bar{a} + \sqrt{\bar{a}^2 - 4\rho_2 k} \frac{8\rho_2}{4\rho_2} + \text{Re} \sqrt{\left[ -\bar{a} + \sqrt{\bar{a}^2 - 4\rho_2 k} \right]^2 - \frac{k\pi^2}{\rho_1 L^2}} < 0. \]
(3.16)
Case II. $\bar{a}^2 < 4\rho_2 k$.

Then

$$y_{1/2} = \frac{-\bar{a} \pm i \sqrt{4\rho_2 k - \bar{a}^2}}{2\rho_2} = \psi_1 \pm i \psi_2, \quad \psi_j \in \mathbb{R}. \quad (3.17)$$

Let $\xi_j := \frac{k\pi^2}{\rho_1}$, then

$$\sqrt{\frac{y_1^2}{4} - \xi_j} = \alpha + i \beta, \quad \alpha, \beta \in \mathbb{R}, \quad \alpha \geq 0,$$

with

$$\alpha^2 - \beta^2 = \frac{\psi_1^2 - \psi_2^2}{4} - \xi_j, \quad 2\alpha\beta = \frac{\psi_1\psi_2}{2}$$

implying

$$\alpha^4 - \left(\frac{\psi_1^2 - \psi_2^2}{4} - \xi_j\right)\alpha^2 - \frac{\psi_1^2 - \psi_2^2}{16} = 0,$$

that is,

$$\alpha = \sqrt{\frac{1}{2} \left(\frac{\psi_1^2 - \psi_2^2}{4} - \xi_j\right) + \sqrt{\frac{1}{4} \left[\frac{\psi_1^2 - \psi_2^2}{4} - \xi_j\right]^2 + \frac{\psi_1^2 - \psi_2^2}{16}}}.$$  \hspace{1cm} (3.18)

Observing

$$\psi_1^2 - 4^2 = \frac{\bar{a}^2 - (4\rho_2 k - \bar{a}^2)}{4\rho_2^2}, \quad \psi_1^2\psi_2^2 = \frac{\bar{a}^2}{4\rho_2^2} \left(\frac{4\rho_2 k - \bar{a}^2}{4\rho_2^2}\right),$$

we conclude from (3.14), (3.17) and (3.18)

$$\max_{r=1,2,3,4} \mathbb{R} \lambda_j' = -\frac{\bar{a}^2}{2\rho_2} + \sqrt{\frac{d_1 - \xi_j}{2} + \left(\frac{d_1 - \xi_j}{2}\right)^2 + d_2}, \quad (3.19)$$

where

$$d_1 := \frac{\psi_1^2 - \psi_2^2}{4}, \quad d_2 := \frac{\psi_1^2\psi_2^2}{16}.$$

Since $x \mapsto f(x) := \sqrt{\frac{d_1 - x}{2} + \left(\frac{d_1 - x}{2}\right)^2 + d_2}$ takes for $x \geq \xi_1 = \frac{k\pi^2}{\rho_1 L^2}$ its maximum in $\xi_1$, we have from (3.19)

$$\max_{j \in \mathbb{N}} \max_{r=1,2,3,4} \mathbb{R} \lambda_j' = -\frac{\bar{a}^2}{2\rho_2} + \sqrt{\frac{1}{2} \left(\frac{\bar{a}^2}{2\rho_2} - \frac{4\rho_2 k - \bar{a}^2}{4\rho_2^2} - \frac{k\pi^2}{\rho_1 L^2}\right) + \sqrt{z^2 + \frac{\bar{a}^2}{4\rho_2^2} \left(\frac{4\rho_2 k - \bar{a}^2}{4\rho_2^2}\right)}} < 0. \quad (3.20)$$

Summarizing Cases I, II we conclude from (3.16), (3.20):

**Theorem 3.2.** Assume (1.6). Then

$$0 > \omega_0 (A) = \max_{j \in \mathbb{N}} \max_{r=1,2,3,4} \mathbb{R} \lambda_j'$$

given in (3.16) if $\bar{a}^2 \geq 4\rho_2 k$, and given in (3.20) if $\bar{a}^2 < 4\rho_2 k$, respectively.
Finally, we shall investigate \( \| (\lambda - A)^{-1} \| \) for \( \text{Re}\lambda > \omega_\sigma \) and demonstrate the SDG property. Let \( \lambda \in \mathbb{C}, \text{Re}\lambda \geq \omega_\sigma + \epsilon \) for some \( \epsilon > 0 \). The equation

\[
(\lambda - A)W = F
\]

implies \( W = (\varphi, \lambda \varphi - F^1, \psi, \lambda \psi - F^3)' \) and \( (\varphi, 4) \) solve

\[
\rho_1 \lambda^2 \varphi - k(\varphi_x + \psi)_x = f_1, \tag{3.21}
\]

\[
\rho_2 \lambda^2 \varphi - b \psi_{xx} + k(\varphi_x + \psi) + \bar{a} \lambda \psi = f_2, \tag{3.22}
\]

\[
\varphi(0) = \varphi(L) = \psi_x(0) = \psi_x(L) = 0, \tag{3.23}
\]

where

\[
f_1 := \rho_1 F^2 + \rho_1 \lambda F^1, \quad f_2 := \rho_2 F^4 + \rho_2 \lambda F^3 + \bar{a} F^3. \tag{3.24}
\]

The boundary conditions admit the expansions

\[
\varphi(x) = \sum_{j=1}^{\infty} g_j v_j(x), \quad \psi_x = \sum_{j=1}^{\infty} h_j w_j(x) \tag{3.25}
\]

where

\[
v_j(x) := \sqrt{\frac{2}{L}} \sin(\theta_j x), \quad w_j(x) := \sqrt{\frac{2}{L}} \cos(\theta_j x), \quad \theta_j := \frac{j\pi}{L}. \tag{3.26}
\]

Then we obtain from (3.21), (3.22), (3.25) the relations

\[
(\rho_1 \lambda^2 + k \theta_j^2) g_j + k \theta_j h_j = f_{1,j}, \tag{3.27}
\]

\[
k \theta_j g_j + (\rho_2 \lambda^2 + b \theta_j^2 + \bar{a} \lambda + k) h_j = f_{2,j}, \tag{3.28}
\]

where \( (f_{1,j}) \) and \( (f_{2,j}) \) denote the Fourier coefficients of \( f_1 \) and \( f_2 \), respectively.

We compute

\[
g_j = \frac{f_{1,j}(\rho_2 \lambda^2 + \bar{a} \lambda + b \theta_j^2 + k) - f_{2,j} k \theta_j}{\rho_1 \rho_2 \lambda^2 (y^2 + \frac{a}{\rho_2} y + \frac{k}{\rho_2})}, \tag{3.29}
\]

\[
h_j = \frac{-k \theta_j f_{1,j} + f_{2,j}(\rho_1 \lambda^2 + k \theta_j^2)}{\rho_1 \rho_2 \lambda^2 (y^2 + \frac{a}{\rho_2} y + \frac{k}{\rho_2})}, \tag{3.30}
\]

where we used the transformation as in (3.5)–(3.7).

We have to estimate

\[
\int_{0}^{L} |\varphi_x(y)|^2 \, dy, \quad \int_{0}^{L} |\lambda \varphi(y)|^2 \, dy, \quad \int_{0}^{L} |\varphi_x(y)|^2 \, dy, \quad \int_{0}^{L} |\lambda \varphi(y)|^2 \, dy \tag{3.31}
\]

in terms of \( \| F \|^2_X \).

Rewriting

\[
g_j = \frac{f_{1,j}(\rho_2 \lambda^2 + b \theta_j^2)}{\rho_1 \rho_2 \lambda^2 (y^2 + \frac{a}{\rho_2} y + \frac{k}{\rho_2})} + \frac{f_{1,j}(\bar{a} \lambda + k) - f_{2,j} k \theta_j}{\rho_1 \rho_2 \lambda^2 (y^2 + \frac{a}{\rho_2} y + \frac{k}{\rho_2})} \tag{3.32}
\]

we hence first prove a bound for

\[
I := \frac{\theta_j^2 (\rho_2 \lambda^2 + b \theta_j^2)^2}{|\rho_1 \rho_2 \lambda^2 (y^2 + \frac{a}{\rho_2} y + \frac{k}{\rho_2})|^2}
\]

which is uniform in \( j \) and \( \lambda \), for \( \text{Re}\lambda \geq \omega_\sigma + \epsilon \).
Observing (3.7) and the essential condition (1.6) again, we obtain
\[ \rho_2 \lambda^2 + b \theta_j^2 = \rho_2 \lambda y, \]  
(3.33)

then we have
\[ I = \left| \theta_j^2 y^2 \right| \left| \rho_1 \lambda^2 (y^2 + \frac{a}{\rho_2} y + \frac{k}{\rho_2})^2 \right|^{-1} \]
and since, by (3.7),
\[ \frac{y}{\lambda} = \frac{k \theta_j}{\rho_1 \lambda^2} + 1 \]
implying
\[ \frac{\theta_j^2}{\rho_1 \lambda^2} = \frac{y}{k \lambda} - 1 \]  
(3.34)

we have
\[ I \leq \left| \frac{y^3}{k \cdot \lambda (y^2 + \frac{a}{\rho_2} + \frac{k}{\rho_2})^2} \right| + \left| \frac{y^2}{y^2 + \frac{a}{\rho_2} + \frac{k}{\rho_2}} \right| \equiv I_1 + I_2. \]  
(3.35)

Without loss of generality we assume \(|\lambda| \geq 1\). There is \(R_0 > 0\) such that for \(|y| \geq R_0\) we know
\[ \left| \frac{y^3}{(y^2 + \frac{a}{\rho_2} + \frac{k}{\rho_2})^2} \right| \leq 1 \]  
(3.36)

while in the compact set \(|y| \leq R_0\) the quadratic polynomial in \(y\) in the nominator does not have a zero since \(\text{Re} \lambda \geq \omega_0 + \varepsilon\). Therefore there is a positive \(c = c(\varepsilon)\) such that for \((y) \leq R_0\) we have
\[ \left| \frac{y^3}{(y^2 + \frac{a}{\rho_2} + \frac{k}{\rho_2})^2} \right| \leq c^2(\varepsilon). \]  
(3.37)

Thus we can estimate \(I_1\), and similarly \(I_2\), uniformly to obtain
\[ I \leq c^2(\varepsilon). \]  
(3.38)

The remaining terms in the representation of \(g_j\) in (3.32) are now estimated as follows:
\[ \text{II} := \frac{\theta_j^2 |\bar{\alpha} \lambda + k|^2}{|\rho_1 \rho_2 \lambda^2 (y^2 + \frac{a}{\rho_2} + \frac{k}{\rho_2})^2|} \]
\[ \leq \frac{\bar{\alpha} |\frac{\theta_j}{\lambda}|^2}{|\rho_1 \rho_2|} \left( \frac{1}{|y^2 + \frac{a}{\rho_2} + \frac{k}{\rho_2}|^2} + \frac{k}{|\rho_1 \rho_2 \lambda^2 (y^2 + \frac{a}{\rho_2} + \frac{k}{\rho_2})^2|} \right) \]
\[ \leq \frac{\bar{\alpha} |\frac{\theta_j}{\lambda}|^2}{|\rho_1 \rho_2|} \left( \frac{1}{|y^2 + \frac{a}{\rho_2} + \frac{k}{\rho_2}|^2} + \frac{1}{k} \frac{1}{|y^2 + \frac{a}{\rho_2} + \frac{k}{\rho_2}|^2} + \frac{k}{|\lambda|^2} \right) \]
\[ \leq \text{const.}, \quad (3.39) \]

where we used (3.34),
\[ \text{III} := \frac{\theta_j^2}{|\rho_1 \rho_2 \lambda^2 (y^2 + \frac{a}{\rho_2} y + \frac{k}{\rho_2})^2|} \leq \frac{c}{|\lambda|}, \]  
(3.40)

c denoting as usual a positive constant that may vary from line to line, similarly \(c(\varepsilon)\). The estimates (3.38)–(3.40) imply
$|\theta_j g_j|^2 \leq c^2(\varepsilon)(|f_{1,j}|^2 + |f_{2,j}|^2).$  

(3.41)

The terms $f_1$ and $f_2$ contain $\lambda F^1$ and $\lambda F^3$, respectively. Observing

\[ \lambda F^1_j = \frac{\lambda}{\theta_j} F^1_j = \frac{\lambda}{\theta_j} (\partial_x F^1)_j \]

and the fact that the factor $|\frac{\lambda}{\theta_j}|$ does not affect the reasoning to obtain (3.41), we have proved

\[ \int_0^L |\varphi_x(y)|^2 dy \leq c^2(\varepsilon)\|F\|_{\mathcal{H}}^2 \]

(3.42)

where $c(\varepsilon)$ depends at most on $\varepsilon$, not on $\lambda$ for $\text{Re}\lambda \geq \omega_\sigma + \varepsilon$.

Since $\lambda = \frac{\lambda}{\theta_j}$ we obtain analogously

\[ \int_0^L |\lambda \varphi(y)|^2 dy + \int_0^L |\varphi_x(y)|^2 dy + \int_0^L |\lambda \varphi(y)|^2 dy \leq c^2(\varepsilon)\|F\|_{\mathcal{H}}^2, \]

(3.43)

hence we proved

\[ \exists c(\varepsilon) > 0, \forall \lambda, \text{ Re}\lambda \geq \omega_\sigma + \varepsilon, \forall F \in \mathcal{H}: \| (\lambda - A)^{-1} F \|_{\mathcal{H}} \leq c(\varepsilon)\|F\|_{\mathcal{H}} \]

which implies by [7] or [15].

**Theorem 3.3.** Assume (1.6). Then the SDG property holds for $A, \omega_0(A) = \omega_\sigma(A)$.

### 4. Exponential stability for indefinite damping

We return to the original system (1.1)–(1.4), or (2.1), with an indefinite damping $a = a(x)$. It will be shown that the system is exponential stable if $\|a - \bar{a}\|_{L^2}$ is small enough. Of course, we keep the basic assumptions (1.5) and (1.6), i.e. we assume

\[ \bar{a} = \frac{1}{L} \int_0^L a(y) dy > 0 \]

(4.1)

and

\[ \frac{\rho_1}{k} = \frac{\rho_2}{b}. \]

(4.2)

**Theorem 4.1.** Assume (4.1) and (4.2). Then there is $\tau > 0$ such that if $\|a - \bar{a}\|_{L^2} < \tau$ the system (2.1) is exponentially stable, that is, the energy $E$, defined in (2.3), to the initial boundary value problem (1.1)–(1.4) satisfies

\[ \exists d > 0, \exists C > 0, \forall \tau \geq 0: \quad E(t) \leq C e^{-2d\tau} E(0). \]

**Proof.** Recalling [7,15] again, it suffices to show that for sufficiently small $\tau > 0$ and for $\lambda$ with $\text{Re}\lambda \geq \omega_\sigma + \varepsilon$, for some $\varepsilon > 0$ such that $\omega_\sigma + \varepsilon < 0, (\lambda - A)W = F$ is uniquely solvable for any $F \in \mathcal{H}$, and $\|W\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}$ with a constant $C > 0$ at most depending on $\tau$ and $\varepsilon$. A fixed point argument will be used. To solve

\[ (\lambda - A)W = F \]

is equivalent to solving

\[ (\lambda - A)W = F + (A - A)W = F - (a - \bar{a})BW \]
with
\[ B := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\rho_2 \end{pmatrix}. \]

Let \( \text{Re} \lambda > \omega \sigma \), then \( W \) should satisfy \( W = (\varphi, \lambda \varphi - F^1, \psi, \lambda \psi - F^3)' \) with \( (\varphi, \psi) \) satisfying, cf. (3.21)–(3.24),

\[
\begin{align*}
\rho_1 & \lambda^2 \varphi - k (\varphi_x + \psi)_x = f_1, \\
\rho_2 & \lambda^2 \psi - b \psi_{xx} + k (\varphi_x + \psi) + \tilde{a} \lambda \psi = (\tilde{a} - a) \lambda \psi + f_2,
\end{align*}
\]

(4.3) \( \quad \) (4.4)

where
\[
\begin{align*}
f_1 & := \rho_1 F^2 + \rho_1 \lambda F^1, \\
f_2 & := \rho_2 F^4 + \rho_2 \lambda F^3 + a F^3.
\end{align*}
\]

(4.6)

(4.4) can be rewritten as

\[
\psi_{xx} - \left( \frac{\rho_2 \lambda^2 + \tilde{a} \lambda + k}{b} \right) \psi = \frac{k}{b} \varphi_x + \frac{a - \tilde{a}}{b} \lambda \psi - \frac{1}{b} f_2.
\]

(4.7)

Let \( \mathcal{N}_\alpha (g) \) denote the solution \( v \) to the Neumann problem

\[
v_{xx} - \alpha^2 v = g, \quad v_x (0) = v_x (L) = 0.
\]

This is well defined if \( \alpha^2 \neq -\frac{j^2 \pi^2}{L^2} \), for \( j = 0, 1, 2, \ldots \), which is guaranteed if

\[
\text{Re} \lambda > - \text{Re} \left( -\tilde{a} + \sqrt{\tilde{a}^2 - 4 \rho_1 k} \right) \frac{2}{b} = : z_0.
\]

(4.8)

The sufficiency of (4.8) can be seen from

\[
\alpha^2 = - \frac{j^2 \pi^2}{L^2} \iff \lambda = \frac{-\tilde{a} \pm \sqrt{\tilde{a}^2 - 4 \rho_1 k (k + \frac{bj^2 \pi^2}{L^2})}}{2 \rho_1}.
\]

Thus, (4.7) can be written as

\[
\psi = \mathcal{N}_\alpha \left( \frac{k}{b} \varphi_x + \frac{a - \tilde{a}}{b} \lambda \psi - \frac{1}{b} f_2 \right),
\]

(4.9)

hence (4.3), (4.4) turn into

\[
\begin{align*}
\rho_1 & \lambda^2 \varphi - k (\varphi_x + \psi)_x = f_1, \\
\rho_2 & \lambda^2 \psi - b \psi_{xx} + k (\varphi_x + \psi) + \tilde{a} \lambda \psi = (\tilde{a} - a) \lambda \left\{ \frac{k}{b} \mathcal{N}_\alpha (\varphi_x) + \frac{\lambda}{b} \mathcal{N}_\alpha ((a - \tilde{a}) \psi) - \frac{1}{b} \mathcal{N}_\alpha (f_2) \right\} + f_2.
\end{align*}
\]

(4.10) \( \quad \) (4.11)

For \( (v, w) \) let

\[
G (v, w) := \frac{k}{b} \mathcal{N}_\alpha (v_x) + \frac{\lambda}{b} \mathcal{N}_\alpha ((a - \tilde{a}) w) - \frac{1}{b} \mathcal{N}_\alpha (f_2)
\]

and consider the mapping

\[
P : H^1_0 ((0, L)) \times H^1 ((0, L)) \to H^1_0 ((0, L)) \times H^1 ((0, L)),
\]

\[
(v, w) \mapsto (\varphi, \psi),
\]

defined as solution \( (\varphi, \psi) \) to
\begin{align}
\rho_1 \lambda^2 \varphi - k(\varphi_x + \psi)_x &= f_1, \quad (4.12) \\
\rho_2 \lambda^2 \psi - b \psi_{xx} + k(\varphi_x + \psi) + \bar{a} \lambda \psi &= (\bar{a} - a) \lambda G(v, w) + f_2, \quad (4.13) \\
\varphi(0) = \varphi(L) = \psi(0) = \psi_x(L) = 0 \quad (4.14)
\end{align}

which is well defined since \( \lambda \in \varrho(A) \). As a norm in the space of definition of \( P \) we define

\[ \| (v, w) \|_\lambda^2 := \int_0^L \rho_1 |\lambda v|^2 + \rho_2 |\lambda w|^2 + b|w_x|^2 + k|v_x + w|^2 \, dx. \]

We shall prove that \( P \) has a fixed point \((\varphi, \psi)\) provided \( \|a - \bar{a}\|_{L^2} \) is small enough. This fixed point is also a solution to (4.3)–(4.5), which can be seen as follows: Let \((\varphi, \psi)\) be this fixed point, and let

\[ \hat{\psi} := G(\varphi, \psi) = \frac{k}{b} \mathcal{N}_a(\varphi_x) + \frac{\lambda}{b} \mathcal{N}_a((a - \bar{a}) \psi) - \frac{1}{b} \mathcal{N}_a(f_2), \]

hence
\[ \hat{\psi}_{xx} - a^2 \hat{\psi} = \frac{k}{b} \varphi_x + \frac{\lambda}{b} (a - \bar{a}) \psi - \frac{1}{b} f_2, \]
\[ \hat{\psi}_x(0) = \hat{\psi}_x(L) = 0, \]

implying
\[ \rho_2 \lambda^2 \hat{\psi} - b \hat{\psi}_{xx} + k(\varphi_x + \hat{\psi}) + \bar{a} \lambda \hat{\psi} = \lambda(\bar{a} - a) \psi + f_2. \quad (4.15) \]

Since \((\varphi, \psi)\) is a fixed point of \( P \), we also have
\[ \rho_2 \lambda^2 \psi - b \psi_{xx} + k(\varphi_x + \psi) + \bar{a} \lambda \psi = \lambda(\bar{a} - a) \hat{\psi} + f_2. \quad (4.16) \]

We conclude for the difference \( \Psi := \hat{\psi} - \psi \),

\[ \Psi_{xx} - a^2 \Psi = \frac{\lambda(a - \bar{a})}{b} \psi \]

or
\[ \Psi = \mathcal{N}_a \left( \frac{\lambda(a - \bar{a})}{b} \psi \right). \]

With the estimates for \( \mathcal{N}_a \) to be proved below, we can conclude, with positive constants \( c_1, c_2 \),

\[ |\Psi| \leq c_1 \|a - \bar{a}\|^1 \|\Psi\|_{L^1} \leq c_1 \|a - \bar{a}\|_{L^2} \|\Psi\|_{L^2}, \]

hence

\[ \|\Psi\|_{L^2} \leq c_2 \|a - \bar{a}\|_{L^2} \|\Psi\|_{L^2}, \]

implying \( \Psi = 0 \) if \( \|a - \bar{a}\|_{L^2} < 1/c_2 \).

Now we shall prove that \( P \) is a contraction mapping provided \( \|a - \bar{a}\|_{L^2} \) is small enough. For this purpose let

\[ (\varphi^j, \psi^j) := P(v^j, w^j), \quad j = 1, 2, \]

and
\[ (\varphi, \psi) := (\varphi^1 - \varphi^2, \psi^1 - \psi^2), \quad (v, w) := (v^1 - v^2, w^1 - w^2). \]

Then \((\varphi, \psi), (v, w)\) satisfy (4.12)–(4.14) with \( f_1 = f_2 = 0 \). Multiplying (4.12) and (4.13) by \( \lambda \varphi \) and \( \lambda \psi \), respectively, and integrating we obtain
\[ \rho_1 \lambda \int_0^L |\lambda \varphi|^2 \, dx + k\lambda \int_0^L (\varphi_x + \psi) \overline{\psi_x} \, dx = 0, \]

\[ \rho_2 \lambda \int_0^L |\lambda \psi|^2 \, dx + b\lambda \int_0^L |\psi_x|^2 \, dx + k\lambda \int_0^L (\varphi_x + \psi) \overline{\psi} \, dx + a \int_0^L |\lambda \psi|^2 \, dx = \lambda \int_0^L (\bar{a} - a) G\lambda \psi \, dx. \]

Summing up we get

\[ \text{Re} \lambda \|(\varphi, \psi)\|^2_2 + a \lambda \int_0^L |\lambda \psi|^2 \, dx = \text{Re} \left\{ \lambda \int_0^L (\bar{a} - a) G\lambda \psi \, dx \right\}. \quad (4.17) \]

Multiplying (4.13) by \( \varphi_x + \psi \) and integrating we get

\[ \rho_2 \lambda^2 \int_0^L \varphi (\varphi_x + \psi) \, dx + b \int_0^L \psi (\varphi_x + \psi) \, dx + k \int_0^L |\varphi_x + \psi|^2 \, dx + a \lambda \int_0^L \varphi (\varphi_x + \psi) \, dx \]

\[ = \lambda \int_0^L (\bar{a} - a) G(\varphi_x + \psi) \, dx. \]

Using the assumption (4.2) on the coefficients we have

\[ (\varphi_x + \psi)_x = \frac{\rho_1}{k} \lambda^2 \varphi = \frac{\rho_2}{b} \lambda^2 \overline{\varphi} \]

implying

\[ k \int_0^L |\varphi_x + \psi|^2 \, dx + \rho_2 (\lambda^2 - \overline{\lambda}^2) \int_0^L \varphi (\varphi_x + \psi) \, dx + \rho_2 \lambda^2 \int_0^L |\psi|^2 \, dx + a \lambda \int_0^L \varphi (\varphi_x + \psi) \, dx \]

\[ = \lambda \int_0^L (\bar{a} - a) G(\varphi_x + \psi) \, dx. \]

Then we conclude

\[ \frac{k}{2} \int_0^L |\varphi_x + \psi|^2 \, dx \leq c \left( \int_0^L |\lambda \psi|^2 \, dx + \lambda \int_0^L (\bar{a} - a) G(\varphi_x + \psi) \, dx \right) \quad (4.18) \]

where \( c \) will denote constants at most depending on the coefficients. We also used the fact that it is sufficient to prove everything for \( \lambda \) such that \( \text{Re} \lambda \in [d_0, d_1] \) with some negative \( d_0 \) and some sufficiently large, but fixed \( d_1 \).

Multiplying (4.13) by \( \overline{\psi} \) and integrating we obtain

\[ \rho_2 \lambda^2 \int_0^L |\psi|^2 \, dx + b \int_0^L |\psi_x|^2 \, dx + k \int_0^L (\varphi_x + \psi) \overline{\psi} \, dx + a \lambda \int_0^L |\psi|^2 \, dx = \lambda \int_0^L (\bar{a} - a) G\lambda \overline{\psi} \, dx. \quad (4.19) \]

It is not difficult to show that \( 0 \in \mathcal{R}(A) \), hence

\[ \exists z_1 > 0, \forall \lambda, |\lambda| \leq z_1 : \quad \lambda \in \mathcal{R}(A) \wedge \| (\lambda - \lambda)^{-1} \| \leq c_1. \quad (4.20) \]

That is, we assume in the sequel w.l.o.g. \( |\lambda| \geq z_1 \). Then

\[ \int_0^L |\psi|^2 \leq \frac{1}{z_1^2} \int_0^L |\lambda \psi|^2. \quad (4.21) \]
Combining (4.19), (4.18) and (4.21) we get
\[
\begin{align*}
    b \int_0^L |\psi_x|^2 \, dx & \leq c \left( |\lambda| \int_0^L |(\bar{a} - a)||G\psi| \, dx + |\lambda| \int_0^L |(\bar{a} - a)||G(\varphi_x + \psi)| \, dx + \int_0^L |\lambda|^2 \, dx \right) \\
    + |\lambda|^2 \int_0^L |(\bar{a} - a)||G\psi| \, dx \\
\end{align*}
\]  
(4.22)

Multiplying (4.12) by $\frac{i}{\lambda} \varphi$ and integrating we obtain
\[
\rho_1 \int_0^L |\lambda \varphi|^2 \, dx \leq c \left( \int_0^L |\varphi_x + \psi|^2 \, dx + \int_0^L |\psi|^2 \, dx \right).
\]  
(4.23)

We conclude from (4.18) and (4.23)
\[
\begin{align*}
    k \frac{L}{4} \int_0^L |\varphi_x + \psi|^2 \, dx + \rho_1 k \frac{L}{4c} \int_0^L |\lambda \varphi|^2 \, dx \\
    \leq \left( c \int_0^L |\lambda \varphi|^2 \, dx + \left| \lambda \right| \int_0^L (a - \bar{a})G(\varphi_x + \psi) \, dx \right) + \left| \lambda \right| \int_0^L (a - \bar{a})G\bar{\lambda}\psi \, dx \right). \\
\end{align*}
\]  
(4.24)

Combining (4.22) and (4.24) we get
\[
\begin{align*}
    k \frac{L}{4} \int_0^L |\varphi_x + \psi|^2 \, dx + \rho_1 k \frac{L}{4c} \int_0^L |\lambda \varphi|^2 \, dx + b \int_0^L |\varphi_x|^2 \, dx \\
    \leq \left( c \int_0^L |\lambda \varphi|^2 \, dx + \left| \lambda \right| \int_0^L (a - \bar{a})G(\varphi_x + \psi) \, dx \right) + \left| \lambda \right| \int_0^L (a - \bar{a})G\bar{\lambda}\psi \, dx \right). \\
\end{align*}
\]  
(4.25)

Multiplying (4.25) by $\frac{2c}{\lambda}$ and combining it with (4.17) yields that there exists $\gamma_0 > 0$ such that we have
\[
(\text{Re} \lambda + \gamma_0) \left\| (\varphi, \psi) \right\|_{\lambda}^2 \leq c \left( |\lambda| + |\lambda|^2 \right) \int_0^L |(\bar{a} - a)||G\psi| \, dx + |\lambda| \int_0^L |(\bar{a} - a)||G(\varphi_x + \psi)| \, dx \right).
\]  
(4.26)

It will now be demonstrated that the right-hand side $R$ of (4.26) can be estimated by
\[
|R| \leq c \|a - \bar{a}\|_{L^2} \left\| (\varphi, \psi) \right\|_{\lambda} \left\| (v, w) \right\|_{\lambda}.
\]  
(4.27)

For this purpose we recall that
\[
G(v, w) = \frac{k}{b} N_\alpha(v_x) + \frac{\lambda}{b} N_\alpha((a - \bar{a})w).
\]

We have the representation
\[
N_\alpha(g) = -\frac{1}{\alpha} \frac{\cosh(\alpha x)}{\sinh(\alpha L)} \int_0^L \cosh(\alpha(L - s))g(s) \, ds + \frac{1}{\alpha} \int_0^L \sinh(\alpha(x - s))g(s) \, ds.
\]

Decomposing $\alpha = a_1 + ia_2$ and $\lambda = \gamma + i\eta$ into its real and imaginary part, respectively, we have
\[
\exists \beta > 0, \forall \lambda, \text{ Re} \lambda \in [d_0, d_1]: \quad |a_1| \geq \beta, \quad a_2 = O(\eta) \quad (|\eta| \to \infty)
\]
(cf. similar considerations in [14]). This allows us to conclude that

$$\left| N_\alpha(v_x)(s) \right| \leq c \| v_x \|_{L^2}, \quad |\lambda|^2 \left| N_\alpha((a - \bar{a})w)(s) \right| \leq c \| a - \bar{a} \|_{L^2} \| \lambda w \|_{L^2}.$$ 

Thus

$$\left| \lambda G(v, w)(s) \right| \leq c \left\| (v, w) \right\|_\lambda$$

which implies (4.27). Combining (4.26), (4.27) we get for

$$\text{Re} \lambda > \gamma_0$$

the estimate

$$\left\| (\varphi, \psi) \right\|_\lambda \leq c \| a - \bar{a} \|_{L^2} \left\| (v, w) \right\|_\lambda \leq d \left\| (v, w) \right\|_\lambda$$

for some $d < 1$ provided $\| a - \bar{a} \|_{L^2}$ is small enough. Thus the existing unique fixed point $(\varphi, \psi)$ of $P$ is the unique solution to (4.3)–(4.5), as explained above, and thus yields the unique solution $W$ to $(\lambda - A)W = F$ through

$$W = (\varphi, \lambda \varphi, \psi, \lambda \varphi)' + (0, -F^1, 0, -F^3)'$$

which implies, using (4.29),

$$\left\| (\varphi, \psi) \right\|_\lambda \leq \| W \|_{\mathcal{H}} + \| F \|_{\mathcal{H}}.$$ 

Let $\tilde{W}$ be the solution to $(\lambda - A)\tilde{W} = F$, i.e.

$$\tilde{W} = (\tilde{\varphi}, \lambda \tilde{\varphi}, \tilde{\psi}, \lambda \tilde{\varphi})' + (0, -F^1, 0, -F^3)'$$

with

$$(\tilde{\varphi}, \tilde{\psi}) = P((0, 0)).$$

Then we obtain, using (4.29), (4.30),

$$\| W \|_{\mathcal{H}} - \| \tilde{W} \|_{\mathcal{H}} \leq \| W - \tilde{W} \|_{\mathcal{H}} = \left\| (\varphi, \psi) - (\tilde{\varphi}, \tilde{\psi}) \right\|_\lambda = \left\| P((\varphi, \psi)) - P((\tilde{\varphi}, \tilde{\psi})) \right\|_\lambda$$

$$\leq d \left\| (\varphi, \psi) - (\tilde{\varphi}, \tilde{\psi}) \right\|_\lambda \leq d \| W \|_{\mathcal{H}} + d \| F \|_{\mathcal{H}},$$

hence

$$\| W \|_{\mathcal{H}} \leq \frac{1}{1 - d} \| \tilde{W} \|_{\mathcal{H}} + \frac{d}{1 - d} \| F \|_{\mathcal{H}},$$

where we used

$$\| \tilde{W} \|_{\mathcal{H}} \leq c \| F \|_{\mathcal{H}}$$

which is justified since $\lambda \in \varrho(A)$. Thus we have proved that for

$$\text{Re} \lambda > \max\{-\gamma_0, z_0\}$$

where $z_0$ is given in (4.8) and $\gamma_0$ is given through (4.26), we have $\lambda \in \varrho(A)$ and the norm of the inverse $(\lambda - A)^{-1}$ is uniformly bounded in $\lambda$. This completes the proof of Theorem 4.1. \hfill \Box

We remark that $\bar{a} = \frac{1}{L} \int_0^L a(x) \, dx$ could be replaced by any positive, fixed $\bar{a}$ yielding a result for a situation near an exponentially stable situation ($a(x) \equiv \bar{a}$). But since $\bar{a}$ depends on $a$ and tends to zero as $\| a \|_{L^1}$ tends to zero (in particular if $\| a \|_{L^\infty} \to 0$), our result is not just a perturbation result, because in the case $a = 0$ there is no energy decay.
References