The Thermoelastic and Viscoelastic Contact of Two Rods

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We consider the thermoelastic and viscoelastic contact problem of two rods and prove the existence of a weak solution using a penalization method and compensated compactness. Moreover, for the thermoelastic contact we show that the weak solution converges to zero exponentially as time goes to infinity, and for the viscoelastic contact we prove that the weak solution decays to zero with the same rates as the relaxation functions do.

1. INTRODUCTION

In this paper we study the existence and the asymptotic behavior of weak solutions of the thermoelastic and viscoelastic contact problem of two rods.

Consider two thin rods, each of which is clamped at one end but may come into contact at their free ends. Assuming that the process is independent of all but the horizontal variable, we can describe the reference configuration of the left rod as $0 \leq x \leq l_1$ and of the right rod as $l_2 \leq x \leq l_3$. The contact condition is given by $v^1 + v^2 = 0$ for $l_1 \leq x \leq l_2$.

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1 (0 < l_1 ≤ l_2), both in nondimensional units. The ends x = 0, 1 are fixed while the ends x = l_1, l_2 are free to expand or contact. Denote g := l_2 – l_1. Let u, θ and v, φ denote the displacement, the temperature difference of the left and right rods, respectively. Then the (linear) motion of the rods is described by the momentum and energy equations in linear thermoelasticity [5, 8]

\[ u_{tt} - a_1 u_{xx} + b_1 \theta_x = 0 \quad \text{in} \ (0, l_1) \times (0, T), \quad (1.1) \]

\[ \theta_t - d_1 \theta_{xx} + b_2 u_{tt} = 0 \quad \text{in} \ (0, l_1) \times (0, T), \quad (1.2) \]

\[ v_{tt} - a_2 v_{xx} + b_2 \phi_x = 0 \quad \text{in} \ (l_2, 1) \times (0, T), \quad (1.3) \]

\[ \phi_t - d_2 \phi_{xx} + b_2 \omega_{xx} = 0 \quad \text{in} \ (l_2, 1) \times (0, T), \quad (1.4) \]

where \( a_i, b_i, d_i \) (i = 1, 2) are constant satisfying \( a_i, d_i > 0, b_i \neq 0 \). The system (1.1)–(1.4) is obtained using the transformations \( x = \hat{x}/L, \theta = \nu_1 \theta, \phi = \nu_2 \hat{\phi} \), where \( \hat{x} \) is the space coordinate, L is the total length of the system, \( \theta \) and \( \phi \) are the temperature differences of two rods, \( \nu_1 \) and \( \nu_2 \) are suitable constants. The initial conditions for (1.1)–(1.4) are given by

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_t(x), \]

\[ \theta(x, 0) = \theta_1(x), \quad x \in [0, l_1] \quad (1.5) \]

\[ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_t(x), \]

\[ \phi(x, 0) = \phi_0(x), \quad x \in [l_2, 1]. \quad (1.6) \]

The boundary conditions at \( x = 0, 1 \) are given by

\[ u(0, t) = \theta_1(0, t) = 0, \quad v(1, t) = \phi_2(1, t) = 0, \quad t \in [0, T]. \quad (1.7) \]

At the free ends the stresses of both rods are zero when there is no contact and equal when there is contact. In both cases the stresses at the free ends are equal. For the mechanical contact at the free ends we consider Signorini’s contact conditions:

\[ \sigma(t) := a_1 u_x(l_1, t) - b_1 \theta(l_1, t) = a_2 v_x(l_2, t) - b_2 \phi(l_2, t), \]

\[ u(l_2, t) \leq g + v(l_2, t), \quad \sigma(t) \leq 0, \]

\[ \sigma(t)[g + v(l_2, t) - u(l_1, t)] = 0, \quad t \in [0, T]. \quad (1.8) \]

(1.8)_2 means that at the free ends, no penetration occurs and the stress is compressive in both rods and is zero when there is no contact. The heat
exchange at the free ends is described by

\[
\begin{align*}
    d_1 \theta_1(l_1, t) &= -\kappa_1 [\theta_1(l_1, t) - \alpha_1 \varphi_1(l_1, t)], \\
    d_2 \varphi_1(l_2, t) &= -\kappa_2 [\alpha_2 \theta_2(l_2, t) - \varphi_2(l_2, t)], \quad t \in [0, T],
\end{align*}
\]

where \(\kappa_1, \kappa_2, \alpha_1, \alpha_2\) are given positive constants. (1.9) says that the heat flux in both rods is proportional to the temperature difference of the free edges. It should be pointed out here that if we take \(\alpha_3 = \alpha_2 = 1\), then the boundary conditions (1.9) are the same as (2.18)–(2.19) in [3]. Therefore the boundary conditions (1.9) are more general than those in [3].

In the second part of the paper we study the contact problem for the case that the two rods are viscoelastic of memory type (the viscoelastic contact problem)

\[
\begin{align*}
    u_{tt} - u_{xx} + \int_0^t \lambda(t - \tau) u_{xx}(\cdot, \tau) \, d\tau &= 0 \quad \text{in } (0, l_1) \times (0, T), \\
    v_{tt} - v_{xx} + \int_0^t h(t - \tau) v_{xx}(\cdot, \tau) \, d\tau &= 0 \quad \text{in } (l_2, 1) \times (0, T),
\end{align*}
\]

together with the initial conditions

\[
\begin{align*}
    (u, u_t)(x, 0) &= (u_0, u_1)(x), \quad x \in [0, l_1], \\
    (v, v_t)(0, x) &= (v_0, v_1)(x), \quad x \in [l_2, 1],
\end{align*}
\]

and the boundary conditions at \(x = 0, 1\),

\[
    u(0, t) = 0, \quad v(1, t) = 0, \quad t \in (0, T),
\]

and Signorini’s contact conditions in the contact point

\[
\begin{align*}
    \sigma(t) &= u_x(l_1, t) - \int_0^t \lambda(t - \tau) u_x(l_1, \tau) \, d\tau \\
    &= v_x(l_2, t) - \int_0^t h(t - \tau) v_x(l_2, \tau) \, d\tau,
\end{align*}
\]
where $\lambda$ and $h$ are the relaxation functions characterizing two viscoelastic rods, the history of $u$ and $v$ are assumed to be zero for $t < 0$.

For the above two-rod contact problems there are only few results concerning existence and stability. Barber and Zhang [4] used linear stability analysis and numerical simulations to examine the transient behavior. Andrews et al. [2] proved the existence of strong solutions to (1.1)–(1.9) in the quasistatic case by reformulating the original problem to a parabolic system containing only the two temperatures. Recently, Andrews et al. [3] obtained the existence of a weak solution of the dynamic problem (1.1)–(1.6) with different boundary conditions by solving a penalized problem and passing to the limit. No time-asymptotic behavior of solutions, however, was obtained in [2, 3].

Our aim in this paper is to establish the asymptotic behavior and the existence of a weak solution to (1.1)–(1.9) and (1.10)–(1.14). We will prove that for (1.1)–(1.9) the weak solution decays exponentially, and for (1.10)–(1.14) the weak solution decays with the same rates as the relaxation functions $\lambda, h$ do. To our knowledge the present paper is the first attempt to investigate the large-time behavior of solutions for dynamic contact problems of two rods in thermoelasticity and viscoelasticity, and the existence of a weak solution of the problem (1.10)–(1.14). We wish to mention that there is a rich mathematical literature devoted to similar problems with simpler geometry settings which involve only a single displacement and/or a single temperature (see [1, 6, 9, 10, 12, 14, 20] for recent results; also see the references cited in [1–3]). Here our paper is concerned with a more complicated situation involving two displacements and temperatures that are coupled not only by a nonlinear system of equations but also in the boundary conditions. In order to prove the existence of a weak solution of (1.1)–(1.9) and (1.10)–(1.14), similarly as in [3, 10, 13], we first consider the variational formulation of the problems, and then solve it by studying a related penalized problem. Finally, using compensated compactness we are able to pass to the limit. To show uniform decay rates (the most difficult part of our analysis), the main difficulties arise from ill-behaved boundary terms induced by the boundary conditions and lack of necessary regularity of the weak solution as we can see for the classical initial boundary value problems in thermoelasticity and viscoelasticity (see the survey article [17, 18] and [15, 16] and the references cited therein). The ill-behaved boundary terms cannot be bounded using the standard Sobolev’s imbedding theorem, since they contain the same order derivatives as the energy integral terms do. To
overcome such difficulties we consider the penalized problem with the mollified initial data, introduce delicate (Lyapunov) functionals, and apply a technique from the boundary control (see Lemma 2.3 below). It should be pointed out here that for (1.10)–(1.14), due to the memory effect, we have to employ different techniques from those used for (1.1)–(1.9) to obtain the existence and asymptotic stability. We will introduce different functionals, apply different multipliers, and exploit the dissipative effect induced by the fading memory to get the good energy integral terms for (1.10)–(1.14) (cf. Lemma 3.5, (3.43), (3.47)).

In Section 2 we prove the existence and the exponential decay for (1.1)–(1.9) and in Section 3 we study the existence, the exponential, and the polynomial decay rates for (1.10)–(1.14).

Throughout this paper the same letter $C$ will denote various positive constants which do not depend on $t$ and $x$.

2. THERMOELASTIC CONTACT

As discussed in the Introduction, to prove the existence result we introduce the variational inequality which is equivalent to system (1.1)–(1.9). We first introduce the sets

\[ K_1 := \{ w \in H^1(0, l_1); w(0) = 0 \}, \quad K_2 := \{ z \in H^1(l_2, 1); z(1) = 0 \}, \]

\[ \mathcal{E} := \{ (w, z) \in K_1 \times K_2; w(l_1) \leq z(l_2) + g \}. \]

**Definition 2.1.** We say that $(u, \theta, v, \varphi)$ is a weak solution of (1.1)–(1.9) when

\[ (u, v) \in W^{1, \infty}([0, T], L^2(0, l_1) \times L^2(l_2, 1)) \cap L^\infty([0, T], \mathcal{E}), \]

\[ (\theta, \varphi) \in L^\infty([0, T], L^2(0, l_1) \times L^2(l_2, 1)) \]

\[ \cap L^2((0, T), H^1(0, l_1) \times H^1(l_2, 1)); \]

\[ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad (2.1) \]

and $(u, \theta, v, \varphi)$ satisfies

\[ \int_0^T \int_0^{l_1} \{ -u_i [\dot{w} - u]\phi_i + a_1u_s(w_s - u_s)\phi - b_1\theta(w_s - u_s)\phi \} \, dx \, dt \]

\[ + \int_0^T \int_0^{l_2} \{ -v_i [\dot{z} - v]\phi_i + a_2v_s(z_s - v_s)\phi - b_2\varphi(z_s - v_s)\phi \} \, dx \, dt \]

\[ \geq -\phi(0) \int_0^{l_1} [w(\cdot, 0) - u_0] \, dx - \phi(0) \int_0^{l_2} [z(\cdot, 0) - v_0] \, dx \]

\[ (2.3) \]
for any $(w, z) \in W^{1,1}((0, T), L^2(0, l_1) \times L^2(l_2, 1)) \cap L^2((0, T), \mathcal{A}), \phi \in C^0([0, T])$ with $\phi(T) = 0$;

$$-(\theta_0, \psi(\cdot, 0)) + \int_0^T \int_0^{l_1} \{ -\theta \psi_t + d_1 \theta \psi_x + b_1 u_{xx} \psi \} \, dx \, dt$$

$$+ \kappa_1 \int_0^T [\theta(l_1, t) - \alpha_1 \varphi(l_2, t)] \psi(l_1, t) \, dt = 0,$$  \hspace{1cm} (2.4)

$$-(\varphi_0, \eta(\cdot, 0)) + \int_0^T \int_{l_2} \{ -\varphi \eta_t + d_2 \varphi \eta_x + b_2 v_{xx} \eta \} \, dx \, dt$$

$$- \kappa_2 \int_0^T [\alpha_2 \theta(l_1, t) - \varphi(l_2, t)] \eta(l_2, t) \, dt = 0$$  \hspace{1cm} (2.5)

for any $(\psi, \eta) \in C^2([0, T], H^1(0, l_1) \times H^1(l_2, 1))$ with $\psi(\cdot, T) = 0$ and $\eta(\cdot, T) = 0$.

It is not difficult to see that any regular solution of (2.1)–(2.5) is a solution of the system (1.1)–(1.9).

2.1. Existence of Weak Solutions

In this subsection we prove that there exists at least one solution of (2.1)–(2.5). The main result is the following.

**Theorem 2.1.** Assume that $u_0, v_0 \in \mathcal{A}, u_2, \theta_0 \in L^2(0, l_1), v_2, \varphi_0 \in L^2(l_2, 1)$. Then there exists a weak solution to the system (1.1)–(1.9).

To prove Theorem 2.1 we first consider an auxiliary penalized problem and prove a uniform a priori estimate. Then we use the $\text{div} - \text{curl}$ Lemma to obtain the weak solution by a limit procedure. The proof is broken into several steps and will be given at the end of this subsection.

We start with the formulation of the penalized problem

$$u_{tt}^\varepsilon - a_1 u_{xx}^\varepsilon + b_1 \theta_x^\varepsilon = 0 \quad \text{in} \ (0, l_1) \times (0, T),$$  \hspace{1cm} (2.6)

$$\theta_t^\varepsilon - d_1 \theta_{xx}^\varepsilon + b_1 u_{xx}^\varepsilon = 0 \quad \text{in} \ (0, l_1) \times (0, T),$$  \hspace{1cm} (2.7)

$$v_{tt}^\varepsilon - a_2 v_{xx}^\varepsilon + b_2 \varphi_x^\varepsilon = 0 \quad \text{in} \ (l_2, 1) \times (0, T),$$  \hspace{1cm} (2.8)

$$\varphi_t^\varepsilon - d_2 \varphi_{xx}^\varepsilon + b_2 v_{xx}^\varepsilon = 0 \quad \text{in} \ (l_2, 1) \times (0, T),$$  \hspace{1cm} (2.9)

together with the initial conditions

$$u^\varepsilon(x, 0) = u_0^\varepsilon(x), \quad u_x^\varepsilon(x, 0) = u_x^\varepsilon(x),$$

$$\theta^\varepsilon(x, 0) = \theta_x^\varepsilon(x), \quad x \in [0, l_1],$$

$$v^\varepsilon(x, 0) = v_0^\varepsilon(x), \quad v_x^\varepsilon(x, 0) = v_x^\varepsilon(x),$$

$$\varphi^\varepsilon(x, 0) = \varphi_0^\varepsilon(x), \quad x \in [l_2, 1],$$  \hspace{1cm} (2.10)

$$\text{and the boundary conditions}$$

$$u^\varepsilon(0, t) = u_1^\varepsilon(t), \quad u^\varepsilon(l_1, t) = u_{l_1}^\varepsilon(t),$$

$$\theta^\varepsilon(0, t) = \theta_l^\varepsilon(t), \quad \theta^\varepsilon(l_1, t) = \theta_{l_1}^\varepsilon(t),$$

$$v^\varepsilon(0, t) = v_1^\varepsilon(t), \quad v^\varepsilon(l_1, t) = v_{l_1}^\varepsilon(t),$$

$$\varphi^\varepsilon(0, t) = \varphi_1^\varepsilon(t), \quad \varphi^\varepsilon(l_1, t) = \varphi_{l_1}^\varepsilon(t),$$  \hspace{1cm} (2.11)
and the boundary conditions at \( x = 0, 1 \)
\[
  u^\varepsilon(0, t) = 0, \quad \theta^\varepsilon_x(0, t) = 0, \quad v^\varepsilon(1, t) = 0, \quad \varphi^\varepsilon_x(1, t) = 0, \quad t \in [0, T],
\]
(2.12)

and the conditions at the possible contact point \((0 < \varepsilon < 1)\)
\[
  a_2u^\varepsilon_t(l_1, t) - b_2\theta^\varepsilon_x(l_1, t) = \frac{1}{\varepsilon} \left[ g + v^\varepsilon(l_2, t) - u^\varepsilon(l_1, t) \right] - \varepsilon u^\varepsilon_t(l_1, t),
\]
\[
  a_2v^\varepsilon_t(l_2, t) - b_2\varphi^\varepsilon_x(l_2, t) = \frac{1}{\varepsilon} \left[ g + v^\varepsilon(l_2, t) - u^\varepsilon(l_1, t) \right] + \varepsilon v^\varepsilon_t(l_2, t);
\]
\[
  d_1\theta^\varepsilon_x(l_1, t) = -\kappa_1(\theta^\varepsilon(l_1, t) - \alpha_1\varphi^\varepsilon(l_2, t)),
\]
\[
  d_2\varphi^\varepsilon_x(l_2, t) = -\kappa_2(\alpha_2\theta^\varepsilon(l_1, t) - \varphi^\varepsilon(l_2, t)).
\]
(2.13)

**Remark 2.1.** The terms \(-\varepsilon u^\varepsilon_t(l_1, t) \) and \(-\varepsilon v^\varepsilon_t(l_2, t) \) are introduced to get regularity of solutions of (2.6)–(2.13). It should be pointed out that our auxiliary penalized problem is different from that considered in [3].

We have the following existence result for the penalized problem (2.6)–(2.13).

**Theorem 2.2.** Assume that
\[
  u^\varepsilon_0, \theta^\varepsilon_0 \in H^2(0, l_1), \quad u^\varepsilon_t \in H^1(0, l_1),
\]
\[
  v^\varepsilon_0, \varphi^\varepsilon_0 \in H^2(l_2, 1), \quad v^\varepsilon_t \in H^1(l_2, 1),
\]
(2.14)

and \( u^\varepsilon_0, u^\varepsilon_t, \theta^\varepsilon_0, v^\varepsilon_0, v^\varepsilon_t, \varphi^\varepsilon_0 \) are compatible with the boundary conditions (2.12)–(2.13). Then there exists a unique solution \((u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon) \) of (2.6)–(2.13) satisfying
\[
  (\partial^j u^\varepsilon, \partial^j v^\varepsilon) \in L^\infty([0, T], H^{2-j}(0, l_1) \times H^{2-j}(l_2, 1)), \quad j = 0, 1, 2,
\]
\[
  (\partial^j \theta^\varepsilon, \partial^j \varphi^\varepsilon) \in L^\infty([0, T], H^{2-2j}(0, l_1) \times H^{2-2j}(l_2, 1)), \quad j = 0, 1,
\]
\[
  (\theta^\varepsilon, \varphi^\varepsilon) \in L^\infty([0, T], L^2(0, l_1) \times L^2(l_2, 1)).
\]
(2.15)

Moreover, we have
\[
  \mathcal{E}(t, u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon) + \int_0^T \int_0^{l_1} |\theta^\varepsilon|^2 \, dx \, dt + \int_0^T \int_{l_2} |\varphi^\varepsilon|^2 \, dx \, dt
\]
\[
  \leq Ce^{CT} \mathcal{E}(0, u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon)
\]
(2.16)
for all \( t \in [0, T] \), where \( C \) is a positive constant independent of \( \epsilon \) and \( t \), and

\[
\mathcal{E}(t, u^\epsilon, \theta^\epsilon, v^\epsilon, \varphi^\epsilon) := \frac{1}{2} \int_0^t \left( |u_x^\epsilon|^2 + |u_x^\epsilon|^2 + |\theta^\epsilon|^2 \right) \, dx \\
+ \frac{1}{2} \int_0^t \left( |v_x^\epsilon|^2 + |v_x^\epsilon|^2 + |\varphi^\epsilon|^2 \right) \, dx \\
+ \frac{1}{2\epsilon} \left[ g + v^\epsilon(l_2, t) - u^\epsilon(l_1, t) \right]^2.
\] (2.17)

**Proof.** We first prove the uniform estimate (2.16). Let \( (u^\epsilon, \theta^\epsilon, v^\epsilon, \varphi^\epsilon) \) be a solution of (2.6)–(2.13) satisfying (2.15). Multiplying (2.6) resp. (2.7) by \( u_x^\epsilon \) resp. \( \theta_x^\epsilon \), and integrate over \((0, l_1)\) resp. \((l_2, l)\). Integrating by parts with respect to \( x \) and recalling the boundary conditions (2.12)–(2.13), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^{l_1} \left( |u_x^\epsilon|^2 + a_1 |u_x^\epsilon|^2 + |\theta_x^\epsilon|^2 \right) \, dx + d_1 \int_0^{l_1} |\theta_x^\epsilon|^2 \, dx + \epsilon |u_x^\epsilon(l_1, t)|^2 \\
= \frac{1}{\epsilon} \left[ g + v^\epsilon(l_2, t) - u^\epsilon(l_1, t) \right] - u_x^\epsilon(l_1, t) \\
- \kappa_1 \left[ \theta^\epsilon(l_1, t) - \alpha_1 \varphi^\epsilon(l_2, t) \right] \theta^\epsilon(l_1, t).
\] (2.18)

Multiplying (2.8) resp. (2.9) by \( v_x^\epsilon \) resp. \( \varphi^\epsilon \), following similar arguments to those used for (2.18), we infer

\[
\frac{1}{2} \frac{d}{dt} \int_0^{l_2} \left( |v_x^\epsilon|^2 + a_2 |v_x^\epsilon|^2 + |\varphi_x^\epsilon|^2 \right) \, dx + d_2 \int_{l_2}^{l} |\varphi_x^\epsilon|^2 \, dx + \epsilon |v_x^\epsilon(l_2, t)|^2 \\
= -\frac{1}{\epsilon} \left[ g + v^\epsilon(l_2, t) - u^\epsilon(l_1, t) \right] - v_x^\epsilon(l_1, t) \\
+ \kappa_2 \left[ \alpha_2 \theta^\epsilon(l_1, t) - \varphi^\epsilon(l_2, t) \right] \varphi^\epsilon(l_2, t).
\] (2.19)

Adding (2.19) to (2.18), one gets

\[
\frac{d}{dt} \mathcal{E}(t, u^\epsilon, \theta^\epsilon, v^\epsilon, \varphi^\epsilon) + d_1 \int_0^{l_1} |\theta_x^\epsilon|^2 \, dx + d_2 \int_{l_2}^{l} |\varphi_x^\epsilon|^2 \, dx \\
+ \epsilon |u_x^\epsilon(l_1, t)|^2 + \epsilon |v_x^\epsilon(l_2, t)|^2 \\
= -\kappa_1 \left[ \theta^\epsilon(l_1, t) - \alpha_1 \varphi^\epsilon(l_2, t) \right] \theta^\epsilon(l_1, t) \\
+ \kappa_2 \left[ \alpha_2 \theta^\epsilon(l_1, t) - \varphi^\epsilon(l_2, t) \right] \varphi^\epsilon(l_2, t).
\] (2.20)
where we have used the relation \( f f = f f + ( f f ) f f = f f f f \). By virtue of Sobolev's imbedding theorem \( (W^{1,1} \to L^\infty) \), we infer

\[
\kappa_1 \left[ \Theta^e (l_1, t) - \alpha_1 \phi^e (l_2, t) \right] \Theta^e (l_1, t) \\
+ \kappa_2 \left[ \alpha_2 \Theta^e (l_1, t) - \phi^e (l_2, t) \right] \phi^e (l_2, t) \\
\leq C \left( \int_0^1 |\Theta^e|^2 d x + \int_0^1 |\phi^e|^2 d x \right) + \frac{d_1}{2} \int_0^1 |\Theta^e|^2 d x + \frac{d_2}{2} \int_0^1 |\phi^e|^2 d x.
\]

(2.21)

Substituting (2.21) into (2.20), we arrive at

\[
\frac{d}{d t} (t, \theta^e, \eta^e, \bar{\eta}^e, \phi^e) + \frac{d_1}{2} \int_0^1 |\Theta^e|^2 d x + \frac{d_2}{2} \int_0^1 |\phi^e|^2 d x \\
\leq C (t, \theta^e, \eta^e, \bar{\eta}^e, \phi^e).
\]

(2.22)

A pplying Gronwall's inequality to (2.22), we obtain (2.16).

Next we prove the existence. First we differentiate (2.6)-(2.13) with respect to \( t \), consider \( u^e_1, \theta^e_1, \psi^e_1, \phi^e_1 \) as new unknowns and solve the differentiated equations by the Faedo-Galerkin method, then we obtain a solution satisfying (2.16) by integration and a limit process.

Let \( \{ w_i \}_{i \in N}, \{ \psi_i \}_{i \in N}, \{ z_i \}_{i \in N}, \) and \( \{ \xi_i \}_{i \in N} \) be complete orthogonal systems in \( K_3, H^1 (0, l_1), K_2 \) and \( H^1 (l_2, 1) \), respectively. Set

\[
(\tilde{u}^n, \tilde{\theta}^n, \tilde{\bar{\eta}}^n, \tilde{\phi}^n)(t) = \left( \sum_{i=1}^{n} h_i^n(t) w_i, \sum_{i=1}^{n} p_i^n(t) \psi_i, \sum_{i=1}^{n} g_i^n(t) z_i, \sum_{i=1}^{n} q_i^n(t) \xi_i \right),
\]

where \( h_i^n, p_i^n, g_i^n, q_i^n \) are determined by the following system of ordinary differential equations

\[
\int_0^{l_1} \left( \tilde{u}_i^n w_j + a_i \tilde{u}_i^n \partial_x w_j - b_i \tilde{\eta}_i^n \bar{\eta}_i^n w_j \right) d x = - e \tilde{u}_i^n (l_1, t) w_j (l_1) + \frac{1}{e} B(t) w_j (l_1),
\]

(2.23)

\[
\int_0^{l_1} \left( \tilde{\theta}_i^n \psi_j + d_i \tilde{\theta}_i^n \partial_x \psi_j - b_i \tilde{\bar{\eta}}_i^n \bar{\eta}_i^n \psi_j \right) d x = - \kappa_1 \left[ \tilde{\theta}_i^n (l_1, t) - \alpha_1 \tilde{\phi}_i^n (l_2, t) \right] \psi_j (l_1),
\]

(2.24)
\[
\int_0^{l_2} \left( \tilde{u}_n^\varepsilon z_j + a_2 \tilde{\psi}^\varepsilon \partial_z \tilde{\psi}^\varepsilon z_j - b_2 \tilde{\phi}^\varepsilon \partial_z \tilde{\phi}^\varepsilon z_j \right) dx = - \epsilon \tilde{v}_i^\varepsilon (l_2, t) z_i (l_2) - \frac{1}{\epsilon} B(t) z_j (l_2),
\]

(2.25)

\[
\int_0^{l_2} \left( \tilde{\psi}_i^\varepsilon \xi_j + d_2 \tilde{\psi}_i^\varepsilon \partial_z \xi_j - b_2 \tilde{\phi}_i^\varepsilon \xi_j \right) dx = \kappa_2 \left[ \alpha \tilde{\theta}_i^\varepsilon (l_1, t) - \tilde{\phi}_i^\varepsilon (l_2, t) \right] \xi_i (l_1)
\]

(2.26)

with

\[
B(t) := \frac{d}{dt} \left[ g + u_0^\varepsilon (l_2) - \tilde{u}_0^\varepsilon (l_1) + \int_0^t (\tilde{u}_n^\varepsilon (l_2, s) - \tilde{u}_n^\varepsilon (l_1, s)) ds \right]
\]

for \( 1 \leq j \leq n \), together with the initial data

\[
(h_i^0, \partial_i h_i^0, p_i^0, g_i^0, \partial_i g_i^0, q_i^0) (0) = \left( h_i^0, h_i^1, p_i^0, g_i^1, g_i^1, q_i^0 \right) \quad 1 \leq i \leq n.
\]

(2.27)

Here \( h_i^0, h_i^1, p_i^0, g_i^0, g_i^1, q_i^0 \) satisfy

\[
\tilde{u}_i^\varepsilon (0) = \sum_{i=1}^n h_i^0 w_i \to u_1^\varepsilon \text{ in } H^1(0, l_1),
\]

\[
\tilde{v}_i^\varepsilon (0) = \sum_{i=1}^n g_i^0 z_i \to v_1^\varepsilon \text{ in } H^1(l_2, 1),
\]

\[
(\tilde{u}_i^\varepsilon, \tilde{\theta}_i^\varepsilon) (0) = \left( \sum_{i=1}^n h_i^1 w_i, \sum_{i=1}^n p_i^0 \psi_i \right) \to (u_2^\varepsilon, \theta_2^\varepsilon) \text{ in } L^2(0, l_1),
\]

\[
(\tilde{v}_i^\varepsilon, \tilde{\phi}_i^\varepsilon) (0) = \left( \sum_{i=1}^n g_i^1 z_i, \sum_{i=1}^n q_i^0 \xi_i \right) \to (v_2^\varepsilon, \phi_2^\varepsilon) \text{ in } L^2(l_2, 1) \text{ as } n \to \infty,
\]

(2.28)

where

\[
u_2^\varepsilon := a_2 \tilde{\phi}_0^\varepsilon - b_1 \partial_1 \tilde{\phi}_0^\varepsilon, \quad \phi_2^\varepsilon := d_2 \tilde{\phi}_1^\varepsilon - b_2 \partial_1 \tilde{\phi}_1^\varepsilon \in L^2(l_2, 1).
\]

By the theory of systems of ordinary differential equations (2.23)–(2.28) is uniquely solvable.
Note \(|f_i| \leq |f_j|\). So it follows from Poincaré’s inequality that

\[
|B(t)|^2 \leq C\left( |\tilde{u}^n(l_1, t)|^2 + |\tilde{v}^n(l_2, t)|^2 \right) \leq C \int_0^{l_1} |\tilde{u}^n_x|^2 \, dx + C \int_{l_2}^{1} |\tilde{v}^n_x|^2 \, dx.
\]  

(2.29)

Multiply (2.23), (2.24), (2.25), and (2.26) by \(\frac{d}{dt} h^n_i, p^n_i, \frac{d}{dt} g^n_i,\) and \(q^n_i,\) respectively; sum \(j\) from 1 to \(n;\) and add the resulting equations. Using (2.29) and the inequality \(\alpha \beta \leq \varepsilon^{-1} \alpha^2 + \varepsilon \beta^2,\) we obtain (cf. the derivation of (2.18)–(2.20))

\[
\frac{1}{2} \frac{d}{dt} \mathcal{G}_n(t) + d_1 \int_0^{l_1} |\tilde{\theta}^n_x|^2 \, dx + d_2 \int_{l_2}^{1} |\tilde{\varphi}^n_x|^2 \, dx \\
\leq \frac{C}{\varepsilon^3} \int_0^{l_1} |\tilde{u}^n_x|^2 \, dx + \frac{C}{\varepsilon^3} \int_{l_2}^{1} |\tilde{v}^n_x|^2 \, dx \\
- \kappa_1 \left[ \theta^n(l_1, t) - \alpha_1 \varphi^n(l_2, t) \right] \theta^n(l_1, t) \\
+ \kappa_2 \left[ \alpha_2 \theta^n(l_1, t) - \varphi^n(l_2, t) \right] \varphi^n(l_2, t).
\]  

(2.30)

where

\[
\mathcal{G}_n(t) := \int_0^{l_1} \left( |\tilde{u}^n|^2 + a_1 |\tilde{u}^n_x|^2 + |\tilde{\theta}^n|^2 \right) \, dx \\
+ \int_{l_2}^{1} \left( |\tilde{v}^n|^2 + a_2 |\tilde{v}^n_x|^2 + |\tilde{\varphi}^n|^2 \right) \, dx.
\]  

(2.31)

Note that the boundary terms on the right-hand side of (2.30) (e.g., (2.30)_2) can be bounded from above in the same manner as that for (2.21). Therefore, recalling the definition of \(\mathcal{G}_n(t),\) we get from (2.30) that

\[
\frac{d}{dt} \mathcal{G}_n(t) + \frac{d_1}{2} \int_0^{l_1} |\tilde{\theta}^n_x|^2 \, dx + \frac{d_2}{2} \int_{l_2}^{1} |\tilde{\varphi}^n_x|^2 \, dx \leq C \varepsilon^{-3} \mathcal{G}_n(t).
\]

Applying Gronwall’s inequality to the above inequality, we conclude

\[
\mathcal{G}_n(t) + \int_0^T \int_0^{l_1} |\tilde{\theta}^n_x|^2 \, dx \, dt + \int_0^T \int_{l_2}^{1} |\tilde{\varphi}^n_x|^2 \, dx \, dt \\
\leq C \varepsilon^{-3} e^{CT} \mathcal{G}_n(0), \quad \forall t \in [0, T].
\]  

(2.32)
Now define
\[ u^n(x, t) := u^n_0(x) + \int_0^t u^n(x, s) \, ds, \]
\[ \theta^n(x, t) := \theta^n_0(x) + \int_0^t \theta^n(x, s) \, ds, \]
\[ v^n(x, t) := v^n_0(x) + \int_0^t v^n(x, s) \, ds, \]
\[ \phi^n(x, t) := \phi^n_0(x) + \int_0^t \phi^n(x, s) \, ds. \]

Thus by virtue of (2.32) and (2.31), we can extract a subsequence of \((u^n, \theta^n, v^n, \varphi^n)\), still denoted by \((u^n, \theta^n, v^n, \varphi^n)\), such that as \(n \to \infty\),
\begin{align*}
(u^n, u^n_e, u^n_x, u^n_{xx}, \theta^n, \theta^n_e) & \rightharpoonup (u_e, u_e, u_e, u_e, \theta_e, \theta_e) \text{ weakly in } L^\infty((0,T), L^2(0, l_2)), \\
(v^n, v^n_e, v^n_x, v^n_{xx}, \varphi^n, \varphi^n_e) & \rightharpoonup (v_e, v_e, v_e, v_e, \varphi_e, \varphi_e) \text{ weakly in } L^\infty((0,T), L^2(l_2,1)), \\
\theta^n_{xx} & \rightharpoonup \theta^n_{ee} \text{ weakly in } L^2((0,T), L^2(0, l_1)), \\
\varphi^n_{xx} & \rightharpoonup \varphi^n_{ee} \text{ weakly in } L^2((0,T), L^2(l_2,1)).
\end{align*}

(233)

Recalling the definition of \(u^n, \theta^n, v^n, \varphi^n\), we integrate Eq. (2.23) over \((0, t)\) with respect to \(t\) to arrive at
\[ \int_0^t \left[ (u^n_{xx} - \bar{u}^n_{xx}(0)) w_j + a_2(u^n_x - \partial_x u^n_0) \partial_x w_j - b_2(\theta^n - \theta^n_0) \partial_x w_j \right] \, dx \]
\[ = \frac{1}{\epsilon} \left[ \left[ g + v^n(l_2, t) - u^n(l_2, t) \right] - \left[ g + v^n(l_2) - u^n_0(l_2) \right] \right] w_j(l_2) \]
\[ - \epsilon(u^n_0(l_2, t) - \bar{u}^n(l_2, 0)) w_j(l_2). \]

(2.34)

Multiply (2.34) by \(\eta \in L^1(0,T)\) and integrate then over \((0,T)\). Letting \(n \to \infty\), keeping in mind that the initial data \(u^n_0, u^n_x, \theta^n_0, \theta^n_0\) are compatible with (2.13), we make use of (2.33), (2.28) and integration by parts with respect to \(x\) to deduce
\[ \int_0^T \int_0^l (u^n_x \eta + a_2 u^n_x \eta - b_2 \theta^n \eta) \, dx \, dt \]
\[ = - \int_0^T \left[ - \frac{1}{\epsilon} \left[ g + v^n(l_2, t) - u^n(l_2, t) \right] - \epsilon u^n(l_2, t) \right] w(l_2, t) \, dt \]
\[ (2.35) \]
for all \( w \in L^1((0, T), K_1) \). Identify (2.35) together with (2.33) implies that \( u^{\epsilon} \in L^1((0, T), L^2(0, l_1)) \), and (2.15), (2.13), and (2.6) hold. Integrating Eqs. (2.24)–(2.26) over \((0, t)\), following the same procedure as used for (2.33)–(2.35), we see that \( \theta^\epsilon, v^\epsilon, \varphi^\epsilon \) together with \( u^\epsilon \) satisfy (2.15) and the system (2.6)–(2.13). Thus we prove the existence.

Now we show the uniqueness. Suppose that \( (u^\epsilon, \theta^\epsilon, v^\epsilon, \varphi^\epsilon) \) and \( (w^\epsilon, \psi^\epsilon, z^\epsilon, \xi^\epsilon) \) are two solutions of (2.6)–(2.13). Denoting \( (U, \Theta, V, \Phi) := (u^\epsilon - w^\epsilon, \theta^\epsilon - \psi^\epsilon, v - z^\epsilon, \varphi^\epsilon - \xi^\epsilon) \), we see that \( (U, \Theta, V, \Phi) \) satisfies

\[
U'' - a_2U_x + b_2\Theta = 0 \quad \text{in} \ (0, l_1) \times (0, T), \tag{2.36}
\]

\[
\Theta'' - d_1\Theta_x + b_1U'' = 0 \quad \text{in} \ (0, l_1) \times (0, T), \tag{2.37}
\]

\[
V'' - a_2V_x + b_2\Phi = 0 \quad \text{in} \ (l_2, 1) \times (0, T), \tag{2.38}
\]

\[
\Phi'' - d_1\Phi_x + b_2V'' = 0 \quad \text{in} \ (l_2, 1) \times (0, T), \tag{2.39}
\]

together with

\[
U(0, t) = \Theta(0, t) = V(1, t) = \Phi(1, t) = 0,
\]

\[
\Theta_x(l_1, t) = -\kappa_1[\Theta(l_1, t) - \alpha_1\Phi(l_1, t)],
\]

\[
\Phi_x(l_1, t) = -\kappa_2[\alpha_2\Theta(l_1, t) - \Phi(l_1, t)],
\]

\[
a_2U_x(l_1, t) + b_2\Phi(l_1, t) = D - \epsilon U_x(l_1, t),
\]

\[
a_2V_x(l_1, t) + b_2\Phi(l_1, t) = D + \epsilon V_x(l_1, t),
\]

where \( D := e^{-\frac{\alpha}{2}l_2}([g + v^\epsilon(l_2, t) - u^\epsilon(l_2, t)] - [g + z^\epsilon(l_2, t) - w^\epsilon(l_2, t)]_+) \).

Multiplying (2.36), (2.37), (2.38), and (2.39) by \( U' \) in \( L^2(0, l_1) \), \( \Theta \) in \( L^2(0, l_1) \), \( V \) in \( L^2(l_2, 1) \), and \( \Phi \) in \( L^2(l_2, 1) \), respectively, we deduce in the same way as in the derivation of (2.18)–(2.20) that

\[
\frac{d}{dt}F(t) + d_1\int_0^{l_1}\Theta_x^2 \, dx + d_2\int_{l_1}^1\Phi_x^2 \, dx + \epsilon U_x^2(l_1, t) + \epsilon V_x^2(l_2, t)
\]

\[
= -\kappa_1[\Theta(l_1, t) - \alpha_1\Phi(l_1, t)]\Theta(l_1, t)
\]

\[
+ \kappa_2[\alpha_2\Theta(l_1, t) - \Phi(l_1, t)]\Phi(l_1, t)
\]

\[
+ (U_x(l_1, t) - V_x(l_2, t))D,
\]

where \( F(t) := \int_0^{l_1}(U_x^2 + U_x^2 + \Theta^2) \, dx + \int_{l_1}^1(V_x^2 + V_x^2 + \Phi^2) \, dx. \)

Recalling the definition of \( D \), taking into account \( |f - h| \leq |f - h| \), utilizing Poincaré’s inequality, we arrive at

\[
\left| (U_x(l_1, t) - V_x(l_2, t))D \right|
\]

\[
\leq \epsilon(U_x^2(l_1, t) + V_x^2(l_2, t)) + Ce^{-\frac{\alpha}{2}}(U_x^2(l_1, t) + V_x^2(l_2, t))
\]

\[
\leq \epsilon(U_x^2(l_1, t) + V_x^2(l_2, t)) + Ce^{-\frac{\alpha}{2}}\left(\int_0^{l_1}U_x^2 \, dx + \int_{l_1}^1V_x^2 \, dx \right). \tag{2.42}
\]
On the other hand, the first two terms on the right-hand side of (2.41) (e.g., (2.41)_2) can be bounded from above in the same way as that for (2.21). Therefore, from (2.41)–(2.42), we conclude that \( \frac{d}{dt}F(t) \leq C e^{-2} F(t) \). Applying Gronwall’s inequality to this inequality and keeping in mind that \( F(0) = 0 \), we obtain \( F(t) = 0 \) on \([0, T]\), which yields \((u^*, \theta^*, \nu^*, \psi^*) = (w^*, \psi^*, z^*, \xi^*)\). The proof of the theorem is complete.

**Remark 2.2.** With the uniqueness we can continue the solution obtained in Theorem 2.2 to \( T = \infty \).

To prove Theorem 2.1 we need the following lemma which gives bounds of solutions on the boundary.

**Lemma 2.3.** Let \( q \) be a \( C^2[\gamma, \beta] \)-function. Let \( a > 0 \) and \( f \in H^1((0, T), L^2(\gamma, \beta)) \). Then for any solution \( V \) with \( \partial_j V \in L^2((0, T), H^{2-j}(\gamma, \beta)) \) \((j = 0, 1, 2)\) of the equation

\[
V_{tt} - a V_{xx} = f, \quad (2.43)
\]

we have that

\[
- \frac{d}{dt} \int_\gamma^\beta q(x)V_xV_x \, dx = - \frac{q(x)}{2}[ (V_{t}^2(x, t) + a V_{x}^2(x, t)) ]_{x=\gamma}^{x=\beta} + \frac{1}{2} \int_\gamma^\beta q'(x)(V_{t}^2 + a V_{x}^2) \, dx - \int_\gamma^\beta q(x)V_{t} f \, dx.
\]

**Proof.** It is easy to see that by (2.43) and integration by parts,

\[
- \frac{d}{dt} \int_\gamma^\beta q(x)V_xV_x \, dx
= - \int_\gamma^\beta q(x)(V_{tt}V_x + V_{t}V_{xx}) \, dx
= - \int_\gamma^\beta \left( aV_{xx} + f \right) qV_x - \frac{1}{2} q' V_x^2 \right) \, dx - \left( \frac{q(x)}{2} V_{t}^2(x, t) \right)_{x=\gamma}^{x=\beta}.
\]

On the other hand we have by a partial integration that

\[
a \int_\gamma^\beta q(x)V_{xx}V_x \, dx = a \left( \frac{q(x)}{2} V_{t}^2(x, t) \right)_{x=\gamma}^{x=\beta} - \frac{a}{2} \int_\gamma^\beta q' V_x^2 \, dx.
\]

Inserting the above identity into (2.44), we obtain the lemma.
Now we are able to prove Theorem 2.1.

Proof of Theorem 2.1. The idea of the proof is to pass to the limit ($\epsilon \to 0$) for the penalized problem to obtain a weak solution. For the limit procedure the main difficulty is to show the convergence of the quadratic terms in (2.3). To overcome this difficulty we apply the div-curl lemma from the compensated compactness theory and Lemma 2.3.

Let $(f^\varepsilon, u_1^\varepsilon, \theta_0^\varepsilon) \in C_0^\infty(0, l_1)$ and $(h^\varepsilon, v_1^\varepsilon, \varphi_0^\varepsilon) \in C_0^\infty(l_2, 1)$ such that

\[
\begin{align*}
(f^\varepsilon, u_1^\varepsilon, \theta_0^\varepsilon) &\to (\partial_t u_0, u_1, \theta_0) \text{ in } L^2(0, l_1), \\
(h^\varepsilon, v_1^\varepsilon, \varphi_0^\varepsilon) &\to (\partial_t v_0, v_1, \varphi_0) \text{ in } L^2(l_2, 1).
\end{align*}
\]  

(2.45)

Define $u_0^\varepsilon(x) := \int_0^x f^\varepsilon dy$ and $v_0^\varepsilon(x) := \int_0^1 h^\varepsilon dy$. Then it is easy to see that $u_0^\varepsilon, u_1^\varepsilon, \theta_0^\varepsilon, v_0^\varepsilon, v_1^\varepsilon, \varphi_0^\varepsilon$ satisfy (2.12)–(2.13). Moreover, from Poincaré’s inequality and (2.45) we have

\[
\begin{align*}
u_0^\varepsilon &\to u_0 \text{ in } H^1(0, l_1), \quad (u_1^\varepsilon, \theta_0^\varepsilon) \to (u_1, \theta_0) \text{ in } L^2(0, l_1), \\
v_0^\varepsilon &\to v_0 \text{ in } H^1(l_2, 1), \quad (v_1^\varepsilon, \varphi_0^\varepsilon) \to (v_1, \varphi_0) \text{ in } L^2(l_2, 1). \tag{2.46}
\end{align*}
\]

Let $(u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon)$ be the solution of (2.6)–(2.13) obtained in Theorem 2.2. Then by (2.46), (2.16), Poincaré’s inequality, and Rellich’s selection theorem, we can extract a subsequence of $(u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon)$, still denoted by $(u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon)$, such that as $\epsilon \to 0$,

\[
\begin{align*}
(u^\varepsilon, u_1^\varepsilon, u_2^\varepsilon, \theta^\varepsilon) &\to (u, u_1, u_2, \theta) \text{ weakly in } L^\infty((0, T), L^2(0, l_1)), \\
(v^\varepsilon, v_1^\varepsilon, v_2^\varepsilon, \varphi^\varepsilon) &\to (v, v_1, v_2, \varphi) \text{ weakly in } L^\infty((0, T), L^2(l_2, 1)), \\
(\theta^\varepsilon, \varphi^\varepsilon) &\to (\theta, \varphi) \text{ weakly in } L^2((0, T), L^2(0, l_1) \times L^2(l_2, 1)), \\
u^\varepsilon &\to u \text{ in } L^2((0, T) \times (0, l_1)), \quad v^\varepsilon \to v \text{ in } L^2((0, T) \times (l_2, 1)). \tag{2.47}
\end{align*}
\]

From curl–div lemma it easily follows that

\[
\begin{align*}|u_1^\varepsilon|^2 - a_1|u_2^\varepsilon|^2 &\to |u_1|^2 - a_1|u_2|^2, \\
|v_1^\varepsilon|^2 - a_2|v_2^\varepsilon|^2 &\to |v_1|^2 - a_2|v_2|^2 \tag{2.48}
\end{align*}
\]

in the sense of distributions. In fact, with $W^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon), V^\varepsilon := (u_1^\varepsilon, -a_1u_2^\varepsilon), \text{ and } (2.6)$ we easily see that $\operatorname{curl}_x \cdot W^\varepsilon = u_2^\varepsilon - u_2^\varepsilon = 0, \operatorname{div}_y \cdot V^\varepsilon = u_1^\varepsilon - a_1u_2^\varepsilon = h^\varepsilon, \theta^\varepsilon$ is uniformly bounded in $L^2((0, T) \times (0, l_1))$. Therefore, applying the curl–div lemma (see Dacorogna [7], Evans [11]), we obtain (2.48). Relation (2.48) can be shown in the same manner.
Now applying Lemma 2.3 with \( a = a_1, \beta = l_1, \) and \( q \in C^1[0, l_1], q(x) = 1 \) for \( x \in [0, \delta] \) and \( q(x) = 0 \) for \( x \in [l_1 - \delta, l_1] \) \((0 < \delta \text{ small enough})\) to Eq. (2.6), integrating the resulting identity with respect to \( t \) over \([0, T]\), using Cauchy–Schwarz’s inequality and (2.16), we get

\[
\frac{1}{2} \int_0^T |u^e_t(\gamma, t)|^2 + a_1 |u^e_\gamma(\gamma, t)|^2 \, dt
\]

\[
= - \int_\gamma^{\frac{T}{T}} q(x) u^e_t u^e_\gamma dx \bigg|_{t=0}^{t=T}
\]

\[
- \frac{1}{2} \int_0^T \int_\gamma^{\frac{T}{T}} q(x) \{u^e_t|^2 + a_1 |u^e_\gamma|^2\} \, dx \, dt - b_1 \int_0^T \int_\gamma^{\frac{T}{T}} q(x) u^e_\gamma \theta^e \, dx \, dt
\]

\[
\leq C e^{CT} F(0, u^e, \theta^e, \nu^e, \varphi^e) \quad (\gamma \in [0, \delta]).
\]  

(2.49)

Integrating (2.49) with respect to \( \gamma \) over \([0, \delta]\), we conclude that

\[
\int_0^T \int_0^\delta (|u^e_t|^2 + |u^e_\gamma|^2) \, dx \, dt \leq C e^{CT} F(0, u^e, \theta^e, \nu^e, \varphi^e).
\]  

(2.50)

Similarly, again applying Lemma 2.3 with \( a = a_1, \gamma = 0, \) and \( q \in C^1[0, l_1], q(x) = 0 \) for \( x \in [0, \delta] \) and \( q(x) = -1 \) for \( x \in [l_1 - \delta, l_1] \) to Eq. (2.6), we obtain

\[
\int_0^T \int_{l_1-\delta}^{l_1} (|u^e_t|^2 + |u^e_\gamma|^2) \, dx \, dt \leq C e^{CT} F(0, u^e, \theta^e, \nu^e, \varphi^e).
\]  

(2.51)

Denote \( \Psi^e := |u^e_t|^2 - |u^e_\gamma|^2, \Psi := |u^e_t|^2 - |u^e|^2. \) Let \( \rho \in C^0(0, l_1) \) with \( \rho(x) = 1 \) for \( x \in [0, \delta) \cup [l_1 - \delta, l_1), \psi \in C^0(0, T) \) with \( \psi(t) = 1 \) for \( t \in [0, \delta) \cup [T - \delta, T], \) and \( \phi \in C^0[0, T] \) be the same as in (2.3). Then it follows from (2.46)–(2.48), (2.50)–(2.51), and (2.16) that

\[
\int_0^T \int_0^l (\Psi^e - \Psi) \phi \, dx \, dt
\]

\[
= \int_0^T \int_0^l (\Psi^e - \Psi) \phi \{ \rho \psi + (1 - \rho) \} \, dx \, dt
\]

\[
\leq \int_0^T \int_0^l (\Psi^e - \Psi) \rho \phi \psi \, dx \, dt
\]

\[
+ C \left( \int_0^T \int_0^\delta + \int_0^T \int_{l_1-\delta}^{l_1} + \int_0^\delta \int_0^{l_1} + \int_T^{T-\delta} \int_0^{l_1} \right) |\Psi^e - \Psi| \, dx \, dt
\]
\[
\leq \int_0^T \int_{I_2}^1 (\Psi^\epsilon - \Psi) \rho \phi \psi \, dx \, dt \\
+ C \delta e^{CT} (S(0, u^\epsilon, \theta^\epsilon, v^\epsilon, \varphi^\epsilon) + S(0, u, \theta, \varphi)) \to 0
\]
as \(\epsilon \to 0\) and then \(\delta \to 0\). Following a procedure similar to that used for (2.52), we can show that
\[
\int_0^T \int_{I_2}^1 (|v_t^\epsilon|^2 - |v_t^\epsilon|^2 - |v_t|^2 + |v_t|^2) \phi \, dx \, dt \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{2.53}
\]

From the above convergences it is not difficult to see that \((u, \varphi)\) satisfies system (2.4)–(2.5). To verify that \((u, \varphi)\) satisfies (2.3) let us take \((w, z) \in W^{1,1}((0, T), \mathcal{X})\). Multiplying equations (2.6) and (2.8) by \(w - u^\epsilon\) and \(z - v^\epsilon\), respectively, adding the resulting equations, integrating by parts, taking into account that\(y \\(0, T), \mathcal{X})\),

\[
[g + v(l_2, t) - u(l_1, t)] [w(l_2, t) - z(l_2, t) + v(l_2, t) - u(l_1, t)] \geq 0
\]
for \((w, z) \in W^{1,1}((0, T), \mathcal{X})\),

and making use of the convergences (2.46)–(2.47), (2.52)–(2.53), and (2.49) (also the same estimate for \(v^\epsilon\)), we see that \((u, \theta, v, \varphi)\) satisfies (2.1)–(2.5) and \((u, v) \in \mathcal{X}\). Hence \((u, \theta, v, \varphi)\) is a weak solution of (1.1)–(1.9). The proof of Theorem 2.1 is complete. 

2.2. Exponential Decay

In this subsection we show that the weak solution of (1.1)–(1.9) established in Theorem 2.1 decays exponentially provided \(\alpha_1, \alpha_2\) in (2.13) are appropriately small. Let \(\mathcal{E}\) be the same as in (2.16) in Theorem 2.2. Then the main result of this section reads:

**Theorem 2.4.** Let \((u, \theta, v, \varphi)\) be the weak solution of (1.1)–(1.9) established in Theorem 2.1. If
\[
\frac{\alpha_1}{2} \frac{\kappa_1}{\kappa_2} + \frac{\alpha_2}{2} < \min \left\{1, \frac{\kappa_1}{\kappa_2}\right\}, \tag{2.54}
\]
then there exist positive constants \(C\) and \(\gamma\) independent of \(t\), such that
\[
\mathcal{E}(t, u, \theta, v, \varphi) \leq C \mathcal{E}(0, u, \theta, v, \varphi) e^{-\gamma t}, \quad t \geq 0. \tag{2.55}
\]

To prove the theorem first we derive the exponential decay with a rate independent of \(\epsilon\) for the solution of the penalized problem by introducing a suitable Lyapunov functional, then making use of the lower semicontinu-
ity of the $L^2$-norm with respect to the weak-* convergence we obtain the desired result.

Before proving Theorem 2.4, we need two lemmas. Let $(u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon)$ be the solution of (2.6)–(2.13) established in Theorem 2.2. Note that (2.20) still holds. After a straightforward calculation we obtain

\[-\kappa_1 [\theta^\varepsilon(l_1, t) - \alpha_1 \varphi^\varepsilon(l_2, t)] \varphi^\varepsilon(l_2, t) + \kappa_1 [\alpha_2 \theta^\varepsilon(l_1, t) - \varphi^\varepsilon(l_2, t)] \varphi^\varepsilon(l_2, t) \leq \frac{[2 - \alpha_1 \kappa_1 - \alpha_2 \kappa_2]}{2} |\varphi^\varepsilon(l_1, t)|^2
\]
\[-\frac{(2 - \alpha_2 \kappa_2 - \alpha_1 \kappa_1)}{2} |\varphi^\varepsilon(l_2, t)|^2 \leq -C |\varphi^\varepsilon(l_1, t)|^2 + |\varphi^\varepsilon(l_2, t)|^2\]

provided (2.54) is satisfied. Substituting the above inequality into (2.20), we arrive at

\[
\frac{d}{dt} E(t, u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon)
+ C \left( \int_0^1 |\theta^\varepsilon|^2 \, dx + \int_{l_2}^1 |\varphi^\varepsilon|^2 \, dx + |\theta^\varepsilon(l_1, t)|^2 + |\varphi^\varepsilon(l_2, t)|^2 \right) \leq 0
\]

(2.56) for $t \geq 0$. It is easy to see that

\[
\int_0^1 |\theta^\varepsilon|^2 \, dx + \int_{l_2}^1 |\varphi^\varepsilon|^2 \, dx \leq 2 \left( \int_0^1 |\theta^\varepsilon|^2 \, dx + \int_{l_2}^1 |\varphi^\varepsilon|^2 \, dx + |\theta^\varepsilon(l_1, t)|^2 + |\varphi^\varepsilon(l_2, t)|^2 \right).
\]

Multiplying the above inequality by $C/4$ and adding the resulting equation into (2.56), we have

\[
\frac{d}{dt} E(t, u^\varepsilon, \theta^\varepsilon, v^\theta, \varphi^\varepsilon)
+ C_1 \left( \int_0^1 (|\theta^\varepsilon|^2 + |\varphi^\theta|^2) \, dx + \int_{l_2}^1 (|\varphi^\varepsilon|^2 + |\varphi^\varepsilon|^2) \, dx \right) \leq 0, \quad t \geq 0
\]

(2.57) for some constant $C_1 > 0$ independent of $t$ and $\varepsilon$. 
Our next objective is to introduce certain functionals such that they give the terms $-|u^2|_t^2$, $-|v^2|_t^2$. For simplicity we will generally suppress the superscript $\epsilon$ (and denote $(u^\epsilon, \theta^\epsilon, v^\epsilon, \varphi^\epsilon)$ by $(u, \theta, v, \varphi)$) in the calculations of the section that follows.

**Lemma 2.5.** For the solution obtained in Theorem 2.2 we have

$$\frac{2}{b_1} \frac{d}{dt} \int_{l_1}^{l_2} \eta_1 u_1 \, dx \leq \int_{l_1}^{l_2} (\delta u_1^2 - u_1^2) \, dx + \delta u_1^2(l_1, t)$$

$$+ C\delta^{-1} \int_{l_1}^{l_2} (\theta^2 + \theta_1^2) \, dx,$$

(2.58)

$$\frac{2}{b_2} \frac{d}{dt} \int_{l_2}^{l_1} \eta_2 v_1 \, dx \leq \int_{l_2}^{l_1} (\delta v_1^2 - v_1^2) \, dx + \delta v_1^2(l_1, t)$$

$$+ C\delta^{-1} \int_{l_2}^{l_1} (\varphi^2 + \varphi_1^2) \, dx,$$

(2.59)

where $0 < \delta < 1$, $\eta_1(x, t) := \int_x^y \theta(y, t) \, dy$ and $\eta_2(x, t) := -\int_x^y \varphi(y, t) \, dy$.

**Proof.** We will only prove (2.58), and (2.59) can be obtained analogously. Integrating Eq. (2.7) over $(0, \infty)$ and taking the boundary conditions (2.12) into account, one gets $\partial_t \eta_1 - d_\eta \theta_1 + b_1 u_1 = 0$. Using this and Eq. (2.6), integrating by parts, and keeping in mind that $\eta_1(0, t) = 0$, we infer

$$\frac{2}{b_1} \frac{d}{dt} \eta_1 u_1 \, dx \geq \frac{2}{b_1} \int_{l_1}^{l_2} (\partial_t \eta_1 u_1 + \eta_1 u_1') \, dx$$

$$= \frac{2}{b_1} \int_{l_1}^{l_2} (d_1 \theta_1 u_1 - b_1 u_1^2 + a_1 \eta_1 u_{xx} - d_2 \eta_1 \theta_1) \, dx$$

$$= 2 \int_{l_1}^{l_2} \left( \frac{d_1}{b_1} \theta_1 u_1 - u_1^2 - \frac{a_1}{b_1} \theta u_x - \frac{d_1}{b_1} \eta_1 \theta_x \right) \, dx$$

$$+ \frac{2a_1}{b_1} \eta_1(l_1, t) u_x(l_1, t)$$

$$\leq \int_{l_1}^{l_2} (\delta u_1^2 - u_1^2) \, dx + C\delta^{-1} \int_{l_1}^{l_2} (\theta_1^2 + \theta^2) \, dx + \delta u_1^2(l_1, t),$$

from which the inequality (2.58) follows. The proof is complete.

In the following lemma we estimate the boundary terms appearing in Lemma 2.5 using Lemma 2.3.
Lemma 2.6. There is a positive constant $C_2$ independent of $t$ and $\epsilon$, such that

$$
- \frac{d}{dt} \int_{l_1}^{l_1} x \frac{u(x)}{l_1} \, dx \leq - \frac{1}{2} \left\{ u_t^2(l_1, t) + a_1 u_x^2(l_1, t) \right\} \\
+ C_2 \int_{l_1}^{l_1} (u_x + u_x + \theta_x^2) \, dx,
$$

$$
- \frac{d}{dt} \int_{l_2}^{x-1} \frac{1}{(1-l_2)} v_x \, dx \leq - \frac{1}{2} \left\{ v_t^2(l_2, t) + a_2 v_x^2(l_2, t) \right\} \\
+ C_2 \int_{l_2}^{x-1} (v_x + v_x + \varphi_x^2) \, dx.
$$

(2.60)

Proof. Applying Lemma 2.3 with

$$(a, \gamma, \beta, f, g(x)) = \left( a_1, 0, l_1, -b_1 \theta_x, \frac{x}{l_1} \right)$$

resp. $$(a, \gamma, \beta, f, g(x)) = \left( a_2, l_2, 1, -b_2 \varphi_x, \frac{(x-1)}{(1-l_2)} \right)$$

to Eq. (2.6) resp. Eq. (2.8), we obtain (2.60) immediately. \[\square\]

Proof of Theorem 2.4. For simplicity we denote $r(t) := [g + v(l_2, t) - u(l_2, t)]^\top$. Multiplying Eq. (2.6) by $u$ in $L^2(0, l_1)$, recalling (2.12)–(2.13), we integrate by parts and use Poincaré’s inequality to arrive at

$$
\frac{d}{dt} \int_{l_1}^{l_1} u_t \, dx \\
= \int_{l_1}^{l_1} (u_t^2 - a_1 u_x^2 + b_1 \theta_x \, dx + \frac{1}{\epsilon} r(t) u(l_1, t) - \epsilon u_t(l_1, t) u(l_1, t) \\
\leq \int_{l_1}^{l_1} \left( u_t^2 - \frac{a_1}{2} u_x^2 + C \theta^2 \right) \, dx + \frac{1}{\epsilon} r(t) \left[ u(l_1, t) - g \right] + \epsilon u_t^2(l_1, t)
$$

(2.61)

provided $\epsilon$ small enough. Similarly, multiplying Eq. (2.8) by $v$, we get

$$
\frac{d}{dt} \int_{l_2}^{x-1} v_t \, dx \leq \int_{l_2}^{x-1} \left( v_t^2 - \frac{a_2}{2} v_x^2 + C \varphi^2 \right) \, dx - \frac{1}{\epsilon} r(t) v(l_2, t) + \epsilon v_t^2(l_2, t).
$$

(2.62)
Therefore, using (2.58)–(2.62) and choosing 
\[
\delta \leq \frac{1}{\epsilon} \min \left\{ \frac{a_1}{C_2}, \frac{a_2}{C_2}, 1 \right\} \min \left\{ \frac{a_1}{1 + a_1}, \frac{a_2}{1 + a_2} \right\}
\]
with \(C_2\) being the same as in Lemma 2.6, we obtain for \(\epsilon\) small enough that 
\[
\frac{d}{dt} \int_{l_1} \left( \frac{2(1 + a_1)}{b_1} \eta u_i - \frac{a_1 x}{4C_2 l_1} u_i u_x + uu_i \right) dx + \frac{a_1}{8} \int_{l_1} (u_i^2 + u_x^2) dx \\
+ \frac{1}{\epsilon} r(t)^2 + \frac{d}{dt} \int_{l_2} \left( \frac{2(1 + a_2)}{b_2} \eta v_i dx - \frac{a_2(x - 1)}{4C_2(1 - l_2)} v_i v_x + vv_i \right) dx \\
+ \frac{a_2}{8} \int_{l_2} (v_i^2 + v_x^2) dx \\
\leq C_3 \left( \int_{l_0} (\theta^2 + \phi^2) dx + \int_{l_2} (\varphi^2 + \varphi_x^2) dx \right)
\]
(2.63)
for some constant \(C_3\) independent of \(t\) and \(\epsilon\). Now define 
\[
\mathcal{Z}(t) := N\mathcal{E}(t, u, \theta, v, \phi) + \int_{l_1} \left( \frac{2(1 + a_1)}{b_1} \eta u_i - \frac{a_1 x}{4C_2 l_1} u_i u_x + uu_i \right) dx \\
+ \int_{l_2} \left( \frac{2(1 + a_2)}{b_2} \eta v_i dx - \frac{a_2(x - 1)}{4C_2(1 - l_2)} v_i v_x + vv_i \right) dx.
\]
(2.64)
Here \(N \geq 2C_3/C_1\) is an appropriately large number such that 
\[
C^{-1}\mathcal{E}(t, u, \theta, v, \phi) \leq \mathcal{Z}(t) \leq C\mathcal{E}(t, u, \theta, v, \phi),
\]
(2.65)
which easily follows from the definition of \(\mathcal{E}\) and Poincaré's inequality.

Now, multiplying (2.57) by \(N\) and adding the resulting inequality into (2.63), recalling \(N \geq 2C_3/C_1\), we infer that \(\frac{d}{dt}\mathcal{Z}(t) + C\mathcal{E}(t, u, \theta, v, \phi) \leq 0\), which together with (2.65) implies \(\frac{d}{dt}\mathcal{Z}(t) + C\mathcal{Z}(t) \leq 0\). Hence \(\mathcal{Z}(t) \leq \mathcal{Z}(0)e^{-Ct}\), which combined with (2.65) gives (recall here \((u, \theta, v, \phi)\) denotes \((u^e, \theta^e, v^e, \phi^e)\)) 
\[
\mathcal{E}(t, u^e, \theta^e, v^e, \phi^e) \leq C\mathcal{E}(0, u^e, \theta^e, v^e, \phi^e)e^{-t/C}, \quad \forall t \geq 0
\]
for \(\epsilon\) small enough, where \(C\) is a positive constant independent of \(t\) and \(\epsilon\). Thus, Theorem 2.4 follows from the above inequality and the lower semicontinuity of the \(L^2\)-norm with respect to \(w^\ast\) convergence (cf. (2.46)–(2.47)).
3. VISCOELASTIC MATERIALS

In this section we first prove the existence of a weak solution of (1.10)–(1.14). Then we show that the weak solution decays with the same rates as the relaxation functions do.

Throughout this section let $K_1$, $K_2$ and $K$ be the same as in the beginning of Section 2.

3.1. Existence

First we introduce the definition of a weak solution of (1.10)–(1.14).

**Definition 3.1.** We say that $(u, v)$ is a weak solution of (1.10)–(1.14) when

$$(u, v) \in W^{1, \infty}((0, T), L^2(0, l_1) \times L^2(l_2, 1)) \cap L^\infty((0, T), \mathcal{R}),$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x),$$

and $(u, v)$ satisfies

$$\int_0^T \int_{l_2}^1 - u_t [w - u] \phi + u_x (w_x - u_x) \phi$$
$$- (w_x - u_x) \phi \int_0^t \lambda(t - \tau) u_x d\tau \right] dx \, dt$$
$$+ \int_0^T \int_{l_2}^1 - v_t [z - v] \phi + v_x (z_x - v_x) \phi$$
$$- (z_x - v_x) \phi \int_0^t h(t - \tau) v_x d\tau \right] dx \, dt$$
$$\geq - \phi(0) \int_0^1 u_1 [w(0) - u_0] \, dx - \phi(0) \int_{l_2}^1 v_1 [z(0) - v_0] \, dx$$

for any

$$(w, z) \in W^{1, 1}((0, T), L^2(0, l_1) \times L^2(l_2, 1))$$
$$\cap L^1((0, T), \mathcal{R}), \quad \phi \in C^\infty[0, T] \quad \text{with } \phi(T) = 0.$$

The existence result in this subsection is the following.
**Theorem 3.1.** Assume that \((u_0, v_0) \in \mathcal{R}, u_1 \in L^2(0, l_1), v_1 \in L^2(l_2, 1),\) and that \(\lambda, h \in C^2[0, \infty)\) satisfy

\[
\lambda(t), h(t) \geq 0, \lambda'(t), h'(t) \leq 0; \quad 1 - \int_0^\infty \lambda(\tau) \, d\tau, 1 - \int_0^\infty h(\tau) \, d\tau > 0.
\]

Then there exists a weak solution of (1.10)–(1.14)

The proof is similar to that of Theorem 2.1 for the thermoelastic contact problem but different techniques have been used to control the memory terms in the derivation of the a priori estimates. We will give the proof at the end of this subsection.

We start by introducing a penalized problem to the system (1.10)–(1.14) \((\varepsilon > 0)\) small:

\[
u_{\varepsilon}^{i} - u_{\varepsilon}^{i} + \int_{0}^{t} \lambda(t - \tau) u_{\varepsilon}^{i} \, d\tau = 0 \quad \text{in } (0, l_1) \times (0, T), \tag{3.5}
\]

\[
u_{\varepsilon}^{i} - v_{\varepsilon}^{i} + \int_{0}^{t} h(t - \tau) v_{\varepsilon}^{i} \, d\tau = 0 \quad \text{in } (l_2, 1) \times (0, T), \tag{3.6}
\]

\[
(u_{\varepsilon}^{i}, u_{\varepsilon}^{i})(x, 0) = (u_0^{i}, u_1^{i})(x), x \in [0, l_1];
\]

\[
(v_{\varepsilon}^{i}, v_{\varepsilon}^{i})(x, 0) = (v_0^{i}, v_1^{i})(x), x \in [l_2, 1];
\]

\[
u_{\varepsilon}^{i}(0, t) = 0, \nu_{\varepsilon}^{i}(1, t) = 0, \tag{3.7}
\]

\[
u_{\varepsilon}^{i}(l_1, t) - \int_{0}^{t} \lambda(t - \tau) u_{\varepsilon}^{i}(l_1, \tau) \, d\tau
\]

\[
= \frac{1}{\varepsilon} [g + v_{\varepsilon}(l_2, t) - u_{\varepsilon}(l_1, t)] - \varepsilon u_{\varepsilon}(l_1, t),
\]

\[
u_{\varepsilon}^{i}(l_2, t) - \int_{0}^{t} h(t - \tau) v_{\varepsilon}^{i}(l_2, \tau) \, d\tau
\]

\[
= \frac{1}{\varepsilon} [g + v_{\varepsilon}(l_2, t) - u_{\varepsilon}(l_1, t)] + \varepsilon v_{\varepsilon}(l_2, t). \tag{3.8}
\]

To facilitate our calculations we introduce

\[
[h \Box \varphi](t) = \int_{0}^{t} h(t - \tau) |\varphi(t) - \varphi(\tau)|^2 \, d\tau, \eta * v = \int_{0}^{t} \eta(t - \tau) v(\tau) \, d\tau
\]
and
\[ \mathcal{P}(t, u, v) := \frac{1}{2} \int_0^t \left( u_t^2 + \left( 1 - \int_0^t \lambda \, d\tau \right) u_x^2 + \lambda \, \Box u \right) \, dx \]
\[ + \frac{1}{2\varepsilon} \left[ g + v(l_2, t) - u(l_1, t) \right]^{-2} \]
\[ + \frac{1}{2} \int_0^1 \left( v_t^2 + \left( 1 - \int_0^l h \, d\tau \right) v_x^2 + h \, \Box v \right) \, dx. \tag{3.9} \]

Note that the sign of \( g \, \Box \varphi \) depends solely on the sign of the function \( g \). In the calculations that follow we will frequently use the following lemma, which can easily be obtained by differentiating \( \eta \, \Box \varphi - (\int_0^t \eta \, d\tau) \varphi^2 \).

**Lemma 3.2.** Let \( \phi, \eta \) be \( C^1 \)-functions. Then we have
\[ 2[\eta \ast \phi] \varphi' = -\eta(t) |\phi|^2 - \frac{d}{dt} \left( \eta \, \Box \varphi - \left( \int_0^t \eta \, d\tau \right) |\phi|^2 \right) + \eta' \, \Box \varphi. \]

In the following theorem we prove the existence of (strong) solutions of (3.5)-(3.8) and a uniform a priori estimate.

**Theorem 3.3.** Assume that
\[ u_0^e \in H^2(0, l_1), \quad u_1^e \in H^1(0, l_1), \quad v_0^e \in H^2(l_2, 1), \quad v_1^e \in H^1(l_2, 1), \]
\[ (u_0^e, v_0^e, u_1^e, v_1^e) \text{ satisfy the boundary conditions (3.8).} \]
Let \( \lambda, h \) satisfy (3.4). Then there exists a unique solution \((u^e, v^e)\) of (3.5)-(3.8) satisfying
\[ (u_j^e, v_j^e) \in L^\infty((0, T), H^{2-j}(0, l_1) \times H^{2-j}(l_2, 1)), \quad j = 0, 1, 2. \]
Moreover,
\[ \mathcal{P}(t, u^e, v^e) \leq \mathcal{P}(0, u^e, v^e), \quad t \geq 0. \tag{3.11} \]

**Proof.** We first prove (3.11). Multiplying (3.5) by \( u_j^e \) in \( L^2(0, l_1) \), integrating by parts, applying Lemma 3.2, and (3.8), we see that
\[ \frac{1}{2} \frac{d}{dt} \int_0^l \left( u_j^e \right)^2 + \left( 1 - \int_0^l \lambda \, d\tau \right) |u_j^e|^2 + \lambda \, \Box u_j^e \right) \, dx \]
\[ = \frac{1}{2} \lambda' \, \Box u_j^e - \frac{1}{2} \lambda \int_0^l |u_j^e|^2 \, dx \]
\[ + \frac{1}{\varepsilon} \left[ g + v^e(l_2, t) - u^e(l_1, t) \right]^{-2} u_j^e(l_2, t) - \varepsilon |u_j^e(l_1, t)|^2. \tag{3.12} \]
Similarly, multiplying (3.6) by $v^\epsilon$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{I_2} \left[ |v^\epsilon|^2 + \left( 1 - \int_0^t h \, dt \right) |v^\epsilon|^2 + h \, \square v^\epsilon \right] \, dx
\]
\[
= \frac{1}{2} h' \square v^\epsilon - \frac{h(t)}{2} \int_{I_2} |v^\epsilon|^2 \, dx
\]
\[
- \frac{1}{\epsilon} \left[ g + v^\epsilon(l_2, t) - u^\epsilon(l_1, t) \right] - v^\epsilon(l_2, t) - \epsilon |v^\epsilon(l_2, t)|^2.
\]
Adding the above inequality into (3.12), we arrive at
\[
\frac{d}{dt} \mathcal{P}(t, u^\epsilon, v^\epsilon)
\]
\[
= \frac{1}{2} \lambda' \square u^\epsilon - \frac{1}{2} \lambda(t) \int_0^t |u^\epsilon|^2 \, dx + \frac{1}{2} h' \square v^\epsilon - \frac{1}{2} h(t) \int_{I_2} |v^\epsilon|^2 \, dx
\]
\[
- \epsilon |u^\epsilon(l_1, t)|^2 - \epsilon |v^\epsilon(l_2, t)|^2,
\]
(3.13)
which together with (3.4) gives (3.11).

We differentiate (3.5)–(3.6) with respect to $t$, and consider $u_t, v_t$ as new unknowns; then applying the Faedo–Galerkin method, we can obtain the existence and uniqueness by using the same arguments as in the proof of the existence in Theorem 2.2 for the thermoelastic contact (cf. (2.24)–(2.43)). Therefore the details will be omitted here.

Now we are in a position to prove Theorem 3.1.

**Proof of Theorem 3.1.** We choose $u_0^\epsilon, u_1^\epsilon, v_0^\epsilon, v_1^\epsilon$ such that they satisfy the assumptions of Theorem 3.3 and (cf. (2.47))
\[
(u_0^\epsilon, u_1^\epsilon, v_0^\epsilon, v_1^\epsilon) \to (u_0, u_1, v_0, v_1)
\]
\[
in H^1(0, l_1) \times L^2(0, l_1) \times H^1(l_2, 1) \times L^2(l_2, 1).
\]
(3.14)

Let $(u^\epsilon, v^\epsilon)$ be the solution of (3.5)–(3.8) obtained in Theorem 3.3. Then by (3.11), Poincaré’s inequality, and Rellich’s selection theorem, we can extract a subsequence of $(u^\epsilon, v^\epsilon)$, still denoted by $(u^\epsilon, v^\epsilon)$, such that as $\epsilon \to 0$,
\[
(u^\epsilon, u^\epsilon, u^\epsilon) \to (u, u, u) \text{ w.-* in } L^\infty((0, T), L^2(0, l_1)),
\]
\[
(v^\epsilon, v^\epsilon, v^\epsilon) \to (v, v, v) \text{ w.-* in } L^\infty((0, T), L^2(l_2, 1)),
\]
\[
u^\epsilon \to u \text{ w.-* in } L^2((0, T) \times (0, l_1)), \quad v^\epsilon \to v \text{ w.-* in } L^2((0, T) \times (l_2, 1)).
\]
(3.15)
Equation (3.5) may be viewed as a linear Volterra integral equation operating on \( u_{xx} \). Upon inverting this operator, we may rewrite (3.5) in the equivalent form (see [19, pp. 149–151])

\[
    u_{xx}^\varepsilon = u_{ol}^\varepsilon - \int_0^t r(t-\tau)u_{ol}^\varepsilon(\tau)\,d\tau = u_{ol}^\varepsilon - r' \ast u_{ol}^\varepsilon - r(0)u_{ol}^\varepsilon + r(t)u_{ol}^\varepsilon,
\]

(3.16)

where \( r \) is the unique solution of the resolvent equation \( r(t) + \int_0^t \lambda(t-\tau)r(\tau)\,d\tau = -\lambda(t), \quad t \geq 0 \). It follows from (3.14) and (3.11) that \( u_{ol}^\varepsilon - u_{xx}^\varepsilon \) is bounded in \( L^\infty([0,T],L^2(0,l_2)) \) uniformly with respect to \( \varepsilon \). Hence, in view of (3.5) we have that

\[
    \lambda \ast u_{xx}^\varepsilon \text{ is bounded in } L^\infty([0,T],L^2(0,l_2)) \text{ uniformly with respect to } \varepsilon.
\]

(3.17)

On the other hand it is easy to see by (3.4) and the relation \( \partial_r(\lambda \ast u_{xx}^\varepsilon) = \lambda(0)u_{xx}^\varepsilon + \lambda' \ast u_{xx}^\varepsilon \) that \( \partial_r(\lambda \ast u_{xx}^\varepsilon) \) is also uniformly bounded in \( L^\infty([0,T],L^2(0,l_2)) \). Therefore by the Rellich selection theorem we can extract a subsequence of \( \lambda \ast u_{xx}^\varepsilon \) such that

\[
    \lim_{\varepsilon \to 0} \int_0^T \int_0^t |\lambda \ast u_{xx}^\varepsilon|^2 \,dx\,dt = \int_0^T \int_0^t |\lambda \ast u_{xx}|^2 \,dx\,dt.
\]

(3.18)

In the same manner we get that

\[
    \lim_{\varepsilon \to 0} \int_0^T \int_0^t |h \ast u_{xx}^\varepsilon|^2 \,dx\,dt = \int_0^T \int_0^t |h \ast u_{xx}|^2 \,dx\,dt.
\]

(3.19)

If we utilize the curl–div lemma, Eqs. (3.5)–(3.6), and Eq. (3.17), we see that \( u^\varepsilon, v^\varepsilon \) still satisfy (2.49) with \( a_1 \) and \( a_2 \) replaced by 1. Now we multiply Eqs. (3.5) and (3.6) by \( w - u^\varepsilon \) and \( z - v^\varepsilon \), respectively, and integrate. We take to the limit as \( \varepsilon \to 0 \), use (3.18)–(3.19), and employ the same arguments as used for (2.49)–(2.53) in Section 2 to see that \( (u,v) \) obtained in (3.15) satisfies (3.1)–(3.3) and \( (u,v) \in \mathbb{R} \). Hence \( (u,v) \) is a weak solution of (1.10)–(1.14).}

3.2. **Exponential Stability**

In this subsection we prove that the weak solution of (1.10)–(1.14) obtained in Theorem 3.1 decays exponentially. For this purpose we assume
that
\[ 0 < \lambda, \, h \in C^3, \quad -c_0 \lambda(t) \leq \lambda'(t) \leq -c_1 \lambda(t), \]
\[ |\lambda''(t)|, |\lambda'''(t)| \leq C \lambda(t), \]
\[ |h''(t)|, |h'''(t)| \leq Ch(t), \]
\[ \bar{\alpha} = 1 - \int_0^\infty \lambda(\tau) \, d\tau, \quad \bar{\beta} = 1 - \int_0^\infty h(\tau) \, d\tau > 0 \quad (3.20) \]
for \( t \geq 0 \) and some positive constants \( c_0, c_1. \)

**Theorem 3.4.** Assume that (3.20) holds. Let \((u, v)\) be the weak solution of (1.10)–(1.14) established in Theorem 3.1. Then there are positive constants \( C, \gamma \) independent of \( t \), such that
\[ \mathcal{F}(t, u, v) \leq C \mathcal{F}(0, u, v) e^{-\gamma t}, \quad t \geq 0. \]

Before going into the proof we need an auxiliary lemma. Let \((u^\epsilon, v^\epsilon)\) be the solution of the penalized problem (3.5)–(3.8) given in Theorem 3.3. For simplicity we will generally suppress the superscript \( \epsilon \) (and denote \((u^\epsilon, v^\epsilon)\) by \((u, v)\)) in the calculations of this subsection that follow. Denoting \( U := u - \lambda^\epsilon u, \, V := v - h^\epsilon v, \) we see, by virtue of (3.5)–(3.8), that \( U, V \) satisfy
\[ U_t - U_{xx} + \lambda'(0)u + \lambda(0)u_{xx} + \lambda'' u = 0, \quad x \in (0, l_1), \, t > 0, \]
\[ V_t - V_{xx} + h'(0)v + h(0)v_{xx} + h'' v = 0, \quad x \in (l_1, 1), \, t > 0; \]
\[ U(x, 0) = u_0(x), \quad U_t(x, 0) = u_t(x) - \lambda(0)u_0(x), \quad x \in [0, l_1], \]
\[ V(x, 0) = v_0(x), \quad V_t(x, 0) = v_t(x) - h(0)v_0(x), \quad x \in [l_1, 1]; \]
\[ (3.23) \]
\[ U(0, t) = 0, \quad V(1, t) = 0, \]
\[ U_t(l_1, t) = \frac{1}{\epsilon} \left[ g + v(l_1, t) - u(l_1, t) \right]^-, \]
\[ -u_t(l_1, t), \]
\[ V_t(l_2, t) = \frac{1}{\epsilon} \left[ g + v(l_2, t) - u(l_1, t) \right]^+, \]
\[ + v_t(l_2, t). \quad (3.24) \]

Now we introduce the new energy functional associated with the system (3.21)–(3.24),
\[ \mathcal{F}(t) := \frac{1}{2} \left\{ \int_0^{l_1} (U_t^2 + U_x^2) \, dx + \int_{l_1}^1 (V_t^2 + V_x^2) \, dx \right\} + \sum_{i=1}^4 S_i(t) + \sum_{i=1}^3 R_i(t), \quad (3.25) \]
where

\[ S_1(t) := \frac{\lambda'(0)}{2} \int_0^{l_2} \left( u^2 + u^2 \int_0^t \lambda \, d\tau - 2(\lambda \ast u) u - \lambda \square u \right) \, dx, \]
\[ S_2(t) := \frac{\lambda(0)}{2} \int_0^{l_1} \left( \lambda' \, \square u - \lambda(0) u^2 - u^2 \int_0^t \lambda' \, d\tau \right) \, dx, \]
\[ S_3(t) := \frac{1}{2} \int_0^{l_1} \left( u^2 \int_0^t \lambda'' \, d\tau - \lambda'' \, \square u - |\lambda' \ast u|^2 \right) \, dx, \]
\[ S_4(t) := \frac{1}{2 \varepsilon} \left[ \|g + v(l_2, t) - u(l_1, t)\|^2 \right], \]
\[ R_1(t) := \frac{h'(0)}{2} \int_{l_2}^{l_1} \left( v^2 + v^2 \int_0^t h \, d\tau - 2(h \ast v) v - h \, \square v \right) \, dx, \]
\[ R_2(t) := \frac{h(0)}{2} \int_{l_2}^{l_1} \left( h' \, \square v - h(0) v^2 - v^2 \int_0^t h' \, d\tau \right) \, dx, \]
\[ R_3(t) := \frac{1}{2} \int_{l_2}^{l_1} \left( v^2 \int_0^t h'' \, d\tau - h'' \, \square v - |\lambda' \ast v|^2 \right) \, dx. \]

The functional \( \mathcal{F} \) will yield the terms \( -\int u_x^2, -\int u_x^2 \) as we will see in the following lemma.

**Lemma 3.5.** We have

\[
\frac{d}{dt} \mathcal{F}(t) \leq C \delta^{-1} \left( \int_0^{l_1} (\lambda(t) u_x^2 + \lambda \square u_x) \, dx + \int_{l_2}^{l_1} (h(t) v_x^2 + h \, \square v_x) \, dx \right)
+ \delta \left( \int_0^{l_1} u_x^2 \, dx + \int_{l_2}^{l_1} v_x^2 \, dx + U_x^2(l_1, t) + V_x^2(l_2, t) \right)
- \lambda(0) \int_0^{l_1} u_x^2 \, dx - h(0) \int_{l_2}^{l_1} v_x^2 \, dx \quad (\forall 0 < \delta < 1). \]

**Proof.** We multiply Eq. (3.21) by \( U_i \) in \( L^2(0, l_2) \) and integrate by parts to deduce that

\[
\frac{1}{2} \frac{d}{dt} \int_0^{l_1} (U_i^2 + U_{x_i}^2) \, dx = -\lambda'(0) \int_0^{l_1} u U_i \, dx - \lambda(0) \int_0^{l_1} u U_i \, dx
- \int_0^{l_1} \lambda'' \ast u U_i \, dx + U_x(l_1, t) U_i(l_2, t) = I_1(t) + I_2(t) + I_3(t) + I_4(t). \]

(3.26)
Next we estimate each term $I_i$. From Lemma 3.2 we obtain after straight-forward calculation that

$$I_1(t) = -\lambda'(0) \int_{l_i}^l \left[ \frac{u^2}{2} - (\lambda \ast u) u \right]_t + (\lambda \ast u) u_t \right] \, dx$$

$$= -\frac{d}{dt} S_1(t) - \frac{\lambda'(0)}{2} \int_{l_i}^l (\lambda'^{\cdot} u - \lambda(t) u^2) \, dx, \quad (3.27)$$

$$I_2(t) = -\lambda(0) \int_{l_i}^1 [u_t^2 - \lambda(0) uu_t - (\lambda' \ast u) u_t] \, dx$$

$$= -\frac{d}{dt} S_2(t) - \lambda(0) \int_{l_i}^1 u_t^2 \, dx + \frac{\lambda(0)}{2} \int_{l_i}^1 [\lambda'' u - \lambda'(t) u^2] \, dx, \quad (3.28)$$

$$I_3(t) = -\int_{l_i}^1 \left[ (\lambda'' \ast u) u_t - \frac{1}{2} \delta_t (\lambda' \ast u)^2 + \lambda'(0)(\lambda' \ast u) u\right.$$  

$$\left. - \lambda(0)(\lambda'' \ast u) u \right] \, dx$$

$$= -\frac{d}{dt} S_3(t) + \int_{l_i}^1 u \int_{l_i}^1 \left[ \lambda'(0) \lambda''(t - \tau) - \lambda'(0) \lambda'(t - \tau) \right]$$

$$\times \left[ (u(\tau) - u(t)) \right] \, d\tau \, dx$$

$$+ \frac{1}{2} \int_{l_i}^1 \left[ 2[\lambda(0) \lambda'(t) - \lambda'(0) \lambda(t)] u^2 - \lambda'' \square u + \lambda''(t) u^2 \right] \, dx$$

$$\leq -\frac{d}{dt} S_3(t) + \delta \int_{l_i}^1 u_t^2 \, dx + C \delta^{-1} \int_{l_i}^1 (\lambda(t) u_t^2 + \lambda \square u_t) \, dx, \quad (3.29)$$

where in (3.29) we have used the inequality $|xy| \leq \delta x^2 + \delta^{-1} y^2 \ (0 < \delta < 1), (3.20)$, and Poincaré’s inequality for $u$. Recalling (3.20) and the boundary condition (3.24), we find that

$$I_4(t) = U_1(l_1, t) u_1(l_1, t) - \lambda(t) U_1(l_1, t) u(l_1, t)$$

$$- U_1(l_1, t) \int_{l_i}^l \lambda'(t - \tau) \{u(l_1, \tau) - u(l_1, t)\} \, d\tau$$

$$\leq \frac{1}{\varepsilon} \left[ g + v(l_1, t) - u(l_1, t) \right] u_1(l_1, t) + \delta U_1^2(l_1, t)$$

$$+ \delta^{-1} \lambda^2(t) u_t^2(l_1, t) + C \delta^{-1} \lambda \square u(l_1, t). \quad (3.30)$$
Combining the estimates (3.27)–(3.30) together and applying Poincaré’s inequality for $u$, we arrive at

$$
\sum_{i=1}^{4} I_i(t) \leq - \sum_{i=1}^{3} \frac{d}{dt} S_i(t) + \frac{1}{\epsilon} \left[ g + v(l_2, t) - u(l_1, t) \right]^{-} u(l_1, t)
$$

$$
- \lambda(0) \int_{0}^{t_1} u_{t}^2 \, dx + \delta \int_{0}^{t_1} u_{x}^2 \, dx + \delta U_{x}^2 (l_1, t)
$$

$$
+ C \delta^{-1} \int_{0}^{t_1} (\lambda(t) u_{x}^2 + \lambda \square u_{x}) \, dx.
$$

(3.31)

Similarly, multiplying (3.22) by $V_t$ in $L^2(l_2, 1)$, following the same procedure as used for (3.26)–(3.31), we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{l_2}^{1} (V_t^2 + V_x^2) \, dx
$$

$$
\leq - \sum_{i=1}^{3} \frac{d}{dt} R_i(t) - \frac{1}{\epsilon} \left[ g + v(l_2, t) - u(l_1, t) \right]^{-} v_i(l_2, t)
$$

$$
- h(0) \int_{0}^{t_1} v_{t}^2 \, dx + \delta \int_{l_2}^{1} v_{x}^2 \, dx + \delta V_{x}^2 (l_2, t)
$$

$$
+ C \delta^{-1} \int_{l_2}^{1} (h(t) v_{x}^2 + h \square v_{x}) \, dx.
$$

(3.32)

Adding (3.32) and (3.31) to (3.26), we obtain the lemma. \[\square\]

It is easy to see that

$$
\int_{0}^{t_1} (U_t^2 + U_x^2) \, dx + \int_{l_2}^{1} (V_t^2 + V_x^2) \, dx \leq C \mathcal{P}(t, u, v),
$$

(3.33)

where $\mathcal{P}(t, u, v)$ is defined by (3.9) in Subsection 3.1. In fact, it follows from Poincaré inequality and (3.20) that

$$
\int_{0}^{t_1} U_t^2 \, dx = \int_{0}^{t_1} \left( u_t - \lambda(t) u - \int_{0}^{t} \lambda'(t - \tau)(u(\tau) - u(t)) \, d\tau \right)^2 \, dx
$$

$$
\leq C \int_{0}^{t_1} (u_t^2 + u_x^2 + \lambda \square u_x) \, dx \leq C \mathcal{P}(t, u, v).
$$

(3.34)

The rest of (3.33) can be shown in the same way.
Recalling the definition of $\mathscr{F}$, taking into account that $\lambda u = \lambda (u - u(t)) + u(i) \int_0^t \lambda \, dt$, utilizing (3.20), Poincaré’s inequality, and (3.33), we get

$$\mathscr{F}(t) \leq C \mathcal{P}(t, u, v), \quad \forall t \geq 0.$$  \hspace{1cm} (3.35)

Now we give the proof of Theorem 3.4.

**Proof of Theorem 3.4.** We first introduce the following functionals in order to estimate the boundary terms appearing in Lemma 3.5:

$$J(t) := \int_0^t u^2 \, dx + \int_{l_2}^1 v^2 \, dx + \frac{\alpha}{2} \left( u^2(l_1, t) + v^2(l_2, t) \right),$$

$$B(t) := - \int_0^t x U u \, dx - \int_{l_2}^1 (x - 1) V v \, dx.$$  \hspace{1cm} (3.36)

Using Eqs. (3.5)–(3.6), integrating by parts with respect to $x$, and applying (3.20) and the inequality $xy \leq \delta x^2 + \delta^{-1} y^2$, we infer

$$\frac{d}{dt} J(t) \leq \int_0^t u^2 \, dx + \int_{l_1}^1 v^2 \, dx - \frac{1}{\varepsilon} \left[ g + v^2(l_2, t) - u^2(l_1, t) \right] \quad \alpha \right)$$

$$- \frac{\tilde{\alpha}}{2} \int_0^t \frac{d}{dt} J(t) \leq - \frac{\beta}{2} \int_{l_1}^1 v^2 \, dx + C \int_0^t \lambda \nabla u \, dx + C \int_{l_2}^1 \nabla v \, dx.$$  \hspace{1cm} (3.37)

Applying Theorem 2.1 in Subsection 2.1 to Eqs. (3.21)–(3.22), we find for $\beta = l_1$, $\gamma = 0$, $q(x) = x$, $f = -\lambda'(0)u - \lambda(0)u_i - \lambda^* u$ and for $\beta = 1$, $\gamma = l_2$, $q(x) = x - 1$, $f = -h'(0)v - h(0)v_i - h^* v$ that

$$\frac{d}{dt} B(t) \leq - \frac{l_1}{2} U^2(l_1, t) - \frac{1 - l_2}{2} V^2(l_2, t)$$

$$+ C \int_0^t \left( u_i^2 + u_i^2 + \lambda \nabla u_i \right) \, dx$$

$$+ C \int_{l_2}^{l_1} \left( v_i^2 + v_i^2 + \lambda \nabla v_i \right) \, dx,$$  \hspace{1cm} (3.38)

where we have also used (3.20), Poincaré inequality, and (3.34).

Recall that the estimate (3.13) still holds. So combining (3.13) and Lemma 3.5 with (3.37)–(3.38), taking $\delta$ in Lemma 3.5 appropriately small (but fixed), and $N$ below sufficiently large, we deduce

$$\frac{d}{dt} \mathcal{P}(t) = \frac{d}{dt} \left[ N \mathcal{P}(t, u, v) + \mathcal{F}(t) + \min\left\{ \lambda(0), h(0) \right\} \left( J(t) + \sqrt{\delta} B(t) \right) \right]$$

$$\leq - \mu_1 \mathcal{P}(t, u, v)$$  \hspace{1cm} (3.39)

for any $t \geq 0$, where $\mu_1$ is a positive constant independent of $t$. 
By virtue of (3.33), (3.35), and Poincaré inequality, we easily see that
there are positive constants $c_3$ and $c_4$, such that
\[ c_3 \mathcal{P}(t, u, v) \leq \mathcal{I}(t) \leq c_4 \mathcal{P}(t, u, v), \quad \forall \ t \geq 0 \] (3.40)
provided $N$ is large enough. Substituting (3.40) into (3.39), one gets
\[ \mathcal{I}'(t) \leq -\left( \mu_1/c_4 \right) \mathcal{I}(t). \]
Therefore $\mathcal{I}(t) \leq \mathcal{I}(0)e^{-\mu_1 t/c_4}$, which together with (3.40) yields
\[ \mathcal{P}(t, u^*, v^*) \leq C \mathcal{P}(0, u^*, v^*)e^{-\mu_1 t/c_4}, \quad \forall \ t \geq 0. \]

From the above inequality, (3.14), and the lower semicontinuity of the $L^2$-norm with respect to $w^*$-convergence, Theorem 3.4 easily follows. The proof is complete. \[ \square \]

### 3.3. Polynomial Decay

In this subsection we prove that when the kernels $\lambda, h$ go to zero with a polynomial rate as $t \to \infty$, then $\mathcal{P}(t, u, v)$ also decays polynomially. We start with introducing the hypotheses on $\lambda$ and $h$: There are constants $p > 2, c_0, c_1 > 0$, such that
\[
0 < \lambda(t), h(t); \quad -c_0 \lambda^{1+ \frac{1}{p}}(t) \leq \lambda'(t) \leq -c_1 \lambda^{1+ \frac{1}{p}}(t),
\]
\[
|\lambda''(t)|, |\lambda'''(t)| \leq c_1 \lambda^{1+ \frac{1}{p}}(t), \quad -c_0 h^{1+ \frac{1}{p}}(t) \leq h'(t) \leq -c_1 h^{1+ \frac{1}{p}}(t);
\]
\[
|h''(t)|, |h'''(t)| \leq c_1 h^{1+ \frac{1}{p}}(t), \quad \forall \ t \geq 0;
\]
\[
0 < 1 - \int_0^\infty \lambda(\tau) \, d\tau, \quad 1 - \int_0^\infty h(\tau) \, d\tau. \tag{3.41}
\]

From the above conditions we easily get
\[ \lambda(t), h(t) \leq C(1 + t)^{-p}, \quad \forall \ t \geq 0. \tag{3.42} \]

The main result of this subsection reads:

**Theorem 3.6.** Let $(u, v)$ be the weak solution of (1.10)–(1.14) established in Theorem 3.1. Assume that $\lambda, h$ satisfy (3.41). Then there is positive constant $C$ independent of $t$, such that
\[ \mathcal{P}(t, u, v) \leq C(1 + t)^{-p}, \quad t \geq 0, \]
where $\mathcal{P}(t, u, v)$ is the same as in (3.9).
Proof. Let \((u^e, v^e)\) be the solution of the penalized problem (3.5)–(3.8) given in Theorem 3.3. We will generally suppress the superscript \(e\) in the calculations that follow. Let \(\mathcal{F}(t), J(t), B(t),\) and \(\mathcal{A}(t)\) be the same as in Subsection 3.2. The first observation is that by Cauchy–Schwarz' inequality,

\[
\int_{\gamma}^{\beta} \left( \int_{0}^{t} \eta(t - \tau) (w(\tau) - w(t)) \, d\tau \right)^2 \, dx \\
\leq \int_{0}^{t} \eta^{1 - \frac{1}{p}}(\tau) \, d\tau \int_{\gamma}^{\beta} \eta^{1 + \frac{1}{p}} \square w \, dx. \tag{3.43}
\]

Utilizing (3.41)–(3.42), following the same procedure as in the proof of Lemma 3.5, we obtain

\[
\frac{d}{dt} \mathcal{F}(t) \leq C \delta^{-1} \left( \int_{0}^{t} \left( \lambda(t) u_x^2 + \lambda^{1 + \frac{1}{p}} \square u_x \right) \, dx \\
+ \int_{l_2}^{t} \left( h(t) v_x^2 + h^{1 + \frac{1}{p}} \square v_x \right) \, dx \right) \\
+ \delta \left( \int_{0}^{t} u_x^2 \, dx + \int_{l_2}^{t} v_x^2 \, dx + U^2(l_1, t) + V^2(l_2, t) \right) \\
- \lambda(0) \int_{0}^{t} u_x^2 \, dx - h(0) \int_{l_2}^{t} v_x^2 \, dx \quad (\forall 0 < \delta < 1). \tag{3.44}
\]

Moreover, by using (3.41)–(3.42), (3.43) with \((\eta, w) = (\lambda, u)\) and \((\eta, w) = (h, v)\), and employing the same arguments as used for (3.37)–(3.39), we infer that

\[
\frac{d}{dt} \mathcal{A}(t) = \frac{d}{dt} \left( N \mathcal{F}(t, u, v) + \mathcal{A}(t) + \frac{\min\{\lambda(0), h(0)\}}{2} J(t) + \sqrt{\delta} B(t) \right) \\
\leq -\mu_2 \left( \int_{0}^{t} \left( u_x^2 + u_x^2 + \lambda^{1 + \frac{1}{p}} \square u_x \right) \, dx \\
+ \left( [g + v(l_2, t) - u(l_1, t)]^2 \\
+ \int_{l_2}^{t} \left( v_x^2 + v_x^2 + h^{1 + \frac{1}{p}} \square v_x \right) \, dx \right) \right) = -\mu_2 \mathcal{A}(t) \tag{3.45}
\]

for \(\delta\) appropriately small and \(N\) sufficiently large, where \(\mu_2\) is a positive constant independent of \(t\).
Integration of (3.13) gives
\[
\int_0^t (u_t^2 + u_x^2) \, dx + \int_{l_2}^1 (v_t^2 + v_x^2) \, dx + \| g + v(l_1, t) - u(l_1, t) \| ^2 \leq C, \quad \forall t \geq 0.
\] (3.46)

By virtue of Hölder’s inequality we have for \( \eta(t) \geq 0 \) the inequality
\[
\int_y^\beta \eta \Box w \, dx \leq \left( \int_y^\beta \eta^{1+r} \frac{1}{p} \Box w \, dx \right)^{(1-r)p} \left( \int_y^\beta \eta' \Box w \, dx \right)^{1/(1-r)p},
\]
\[0 \leq r < 1.\] (3.47)

Note that \( \eta' \Box w \, dx \leq 2 \| \eta'(\tau) \| ds \sup_{[0, t]} |w|^2 \, dx \). So applying (3.47) with \( r = 1 - \frac{1}{p} \) to \( \lambda, u_x \) and \( h, v_x \) respectively, utilizing (3.42) and (3.46), we find that
\[
\int_0^l \lambda \Box u_x + \int_{l_2}^1 h \Box v_x \, dx \leq CY^2(t),
\] (3.48)

where
\[
Y(t) := \int_0^l \lambda \Box u_x + \int_{l_2}^1 h \Box v_x \, dx.
\]

Using the assumption (3.41), it can easily be seen that (3.40) still holds for \( \lambda, h \) satisfying (3.41). Moreover, recalling the definition of \( \zeta(t) \), we have by (3.48), (3.46), and (3.40) that \( \zeta(t) \geq C \mathcal{P}^2(t, u, v) \geq C \mathcal{P}^2(t) \). Hence, from (3.45) we get \( \mathcal{S}'(t) \leq -C \mathcal{P}^2(t) \), which gives \( \mathcal{S}(t) \leq C(1 + t)^{-1} \) for \( t \geq 0 \). This together with (3.40) yields
\[
\mathcal{S}(t, u, v) \leq C(1 + t)^{-1}, \quad t \geq 0.
\] (3.49)

Applying (3.47) with \( r = 1 - \frac{2}{p} \) to \( \lambda, u_x \) and \( h, v_x \) respectively, employing (3.42), (3.46), and (3.49), we arrive at
\[
\int_0^l \lambda \Box u_x + \int_{l_2}^1 h \Box v_x \, dx \leq CY^2(t) \left( \int_0^1 \frac{d\tau}{(1 + t - \tau)^{p-2}(1 + \tau)} \right)^{\frac{1}{2}} \leq CY^2(t),
\] (3.50)

which together with (3.46) and (3.40) yields \( \zeta(t) \geq C \mathcal{P}^2(t, u, v) \geq C \mathcal{P}^2(t) \). So from (3.45) we get \( \mathcal{S}'(t) \leq -C \mathcal{P}^2(t) \). This gives \( \mathcal{S}(t) \leq C(1 + t)^{-2} \), from which and (3.40) it follows that
\[
\mathcal{S}(t, u, v) \leq C(1 + t)^{-2}, \quad t \geq 0.
\] (3.51)
Again applying (3.47) with \( r = 0 \) to \( \lambda, u, \) and \( h, v, \) and using (3.51), we obtain analogously to (3.50) that \( \phi(t) \geq C_{\theta^1 + \delta^1} \tilde{\varphi}(t, u, v) \geq C_{\varphi^1 + \delta^1} \tilde{\varphi}(t). \) Therefore \( \tau(t) \leq -C_{\theta^1 + \delta^1} \tilde{\varphi}(t) \), which implies \( \mathcal{A}(t) \leq C(1 + t)^{-p}. \) In view of (3.40) we conclude \( \mathcal{P}(t, u^*, v^*) \leq C(1 + t)^{-p} \) for all \( t \geq 0 \), where \( C \) is a constant independent of \( t \) and \( \epsilon. \) Thus Theorem 3.6 immediately follows from the lower semicontinuity.

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