Coding of the geodesic flow on hyperbolic surfaces with finite volume

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Notations

- $\mathbb{D} = \left(\{z \in \mathbb{C} \mid |z| < 1\}, \dfrac{2|dz|}{1 - |z|^2} \right)$ is the Poincaré disk.
- $\partial\mathbb{D} = \mathbb{S}^1$ is its boundary at infinty.
- $\Gamma < \mathrm{Isom}(\mathbb{D})$ is a fuchsian group of cofinite volume.
- $M = \mathbb{D}/\Gamma$ is a Riemann surface of finite volume.
- $\varphi^t : T^1M \to T^1M$ is the geodesic flow on M.
- Oriented geometric geodesics of $\mathbb{D}$ can be identified with $\mathbb{T}^2 \setminus \Delta$.

Theorem (A)

There exists

1. *a set* $C \subset \mathbb{T}^2$ *that is the disjoint union of a finite number of rectangles*
2. *a bijection* $T_C : C \to C$ *that can be written*

$$T_C(x, y) = (\gamma_R[y](x), \gamma_R[y](y))$$

where $\gamma_R[.] : \mathbb{S}^1 \to \Gamma$ *only takes a finite number of values*

that is conjugated to a cross section of the geodesic flow φ^t *on* M *(except for a set of measure* 0 *if* M *is not compact). Moreover, that conjugacy* φ *can be written*

$$\varphi(x, y) = (g[x, y](x), g[x, y](y))$$

where $g[.] : \mathbb{T}^2 \to \Gamma$ *only takes a finite number of values and hence can be written as a word in* Γ *of bounded length.*

Theorem (B)

Let

$$\begin{array}{rcll} T_R & : & \mathbb{S}^1 & \longrightarrow \mathbb{S}^1 \\ & & y & \longmapsto \gamma_R[y](y) \end{array}$$

be the (right) Bowen-Series transform. Note $\mathcal{L}_{T_R,s}$ its transfer operator. Then a bounded operator ν in a certain space (described later) satisfies

$$\mathcal{L}^{\star}_{T_R,s}\nu = \nu$$

if and only if ν the Helgason boundary value of some $f : \mathbb{D} \to \mathbb{C}$ such that

$$\Delta f = -s(1-s)f$$

$$\forall \gamma \in \Gamma, f \circ \gamma = \gamma$$

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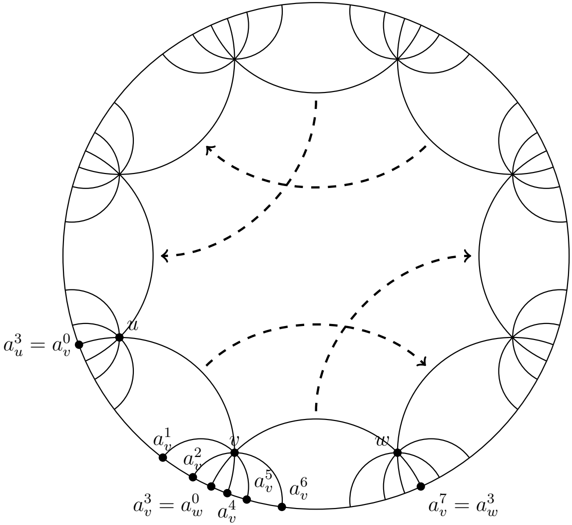
Fundamental domains

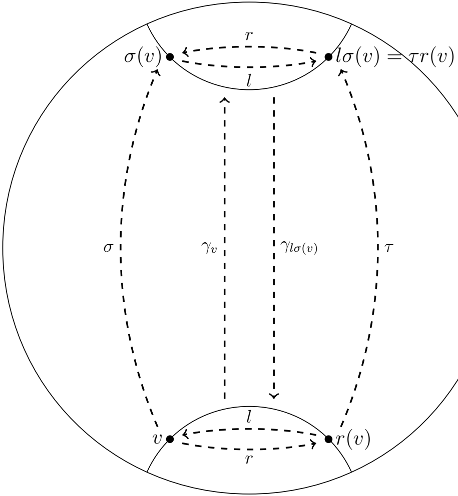
Overview



- convex hyperbolic polygon of finite volume
- sides are paired by isometries
- a side can be paired with itself
- $E = \{ \text{sides} \}$
- $V = \{ \text{vertices} \}$
- γ_v is the isometry for side on the right of v
- $\Gamma = \langle \gamma_v \mid v \in V \rangle$

Fundamental domains

Permutations of vertices



l : $v \mapsto$ first vertice on the left of v

r : $v \mapsto$ first vertice on the right of v

σ : $v \mapsto \gamma_v(v)$

τ : $v \mapsto \gamma_{l(v)}(v)$

$$\tau = \sigma^{-1}$$

$$l\sigma = \tau r$$

Fundamental domains

Even corners fundamental domains

$$\mathcal{N}_v = \begin{cases} \{\gamma(e) \mid e \in E, \gamma \in \Gamma, v \in \gamma(e)\} & \text{if } v \in \mathbb{D} \\ \{\, (v, l(v)), (v, r(v)) \,\} & \text{if } v \in \mathbb{S}^1 \end{cases}$$

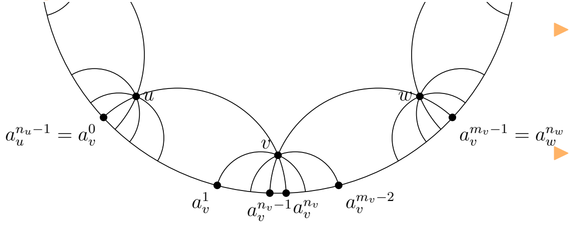
$$\mathcal{N} = \bigcup_{v \in V} \mathcal{N}_v$$

Even corners hypothesis (Series) : No geodesic of $\mathcal{N}$ cuts the interior of the fundamental domain.

For any signature $(g; m_1, \ldots, m_r; s)$, one can construct a fuchsian group of this signature and an *even corners* fundamental domain for this group.

Fundamental domains

Endpoints for an inner vertice



- If $v \in V \cap \mathbb{D}$, $\mathcal{N}_v$ induces $m_v = 2n_v$ points on $\mathbb{S}^1$.
- $v \mapsto n_v$ is constant on the σ-orbits.

Endpoints of $\mathcal{N}_v$ are labelled $a_v^0, \ldots, a_v^{m_v-1}$ from left to right (relatively to the domain).

$$\forall v, \forall k, \gamma_v(a_v^k) = a_{\sigma(v)}^{k+1} \text{ and } \gamma_{l(v)}(a_v^k) = a_{\tau(v)}^{k-1}$$

Proposition

$$\forall v \in V \cap \mathbb{D}, \gamma_{\sigma^{m_v-1}(v)} \ldots \gamma_{\sigma(v)}\gamma_v = id$$

Fundamental domains

Endpoints for a cusp

If $v \in V \cap \mathbb{S}^1$, we artificially set $n_v = 3$ so that

$$a_v^1 = a_v^2 = a_v^3 = a_v^4 = v.$$

The previous formulae no longer hold.

From the geodesic flow to the Bowen-Series coding

Fundamental domains

From the geodesic flow to the Bowen-Series coding

- Geodesic billiard and coding
- Conjugacy between B and C
- The example of a group of signature $(0; 2; 3)$
- The example of the modular surface

Orbit-equivalence property and transfer operator

- Invariance property for relations on the boundary
- Application to the transfer operator eigenproblem

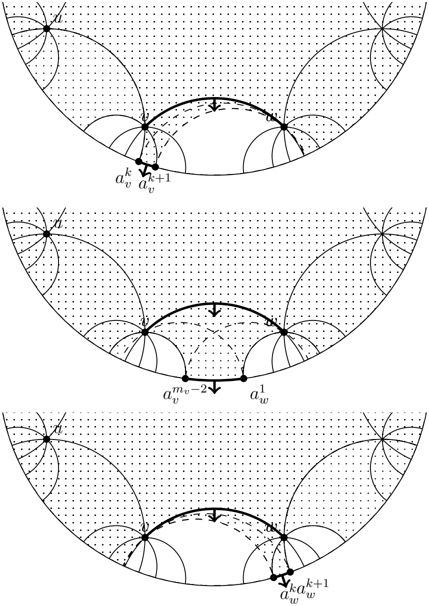
Possible extensions and conclusion

Geodesic billiard

We describe a subset of $\mathbb{T}^2$ that represents all the geodesics of $\mathbb{D}$ that :

- cut the interior of the fundamental domain ;
- pass by a vertice, leaving the fundamental domain on their left.

Geodesic billiard



Geodesic billiard

Geodesic tubes

If $v, w \in \overline{\mathbb{D}}$ and $a, b \in \mathbb{S}^1$,

$$]w; v] \otimes [a; b[= \bigsqcup_{y \in [a;b[}]w \star y; v \star y]$$

where $v \star y$ is the other end of the geodesic starting in y and passing through v.

It is the set of all geodesics that cross $]w; v] \subset \overline{\mathbb{D}}$ and end in $[a; b[\subset \mathbb{S}^1$.

Geodesic billiard

Definition

For $v \in V$ and $w = r(v)$, let $B_v^{-n_v+1}, \dots, B_v^0, \dots, B_v^{n_w-1}$ be :

$$\forall k \in [\![n_v - 1 \,;\, m_v - 3]\!], B_v^{k-m_v+2} =]w; v] \otimes \left[a_v^k; a_v^{k+1}\right[$$
$$B_v^0 =]w; v] \otimes \left[a_v^{m_v-2}; a_w^1\right[$$
$$\forall k \in [\![1 \,;\, n_w - 1]\!], B_v^k =]w; v] \otimes \left[a_w^k; a_w^{k+1}\right[$$

Definition

The *geodesic billiard* is defined by :

$$B = \bigsqcup_{v \in V} \bigsqcup_{k=-n_v+1}^{n_{r(v)}-1} B_v^k$$

$$\forall x \in B_v^k, T_B(x) = \gamma_v(x)$$

Geodesic billiard

Structure

Theorem

$T_B : B \to B$ is a bijection of B such that

$$T_B(x, y) = (\gamma_B[x, y](x), \gamma_B[x, y](y))$$
$$T_B^{-1}(x, y) = (\tilde{\gamma}_B[x, y](x), \tilde{\gamma}_B[x, y](y))$$

where $\gamma_B[x, y], \tilde{\gamma}_B[x, y] \in \Gamma$.

Geodesic billiard

Structure

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where $\gamma_B[x,y], \tilde{\gamma}_B[x,y] \in \Gamma$.

Theorem

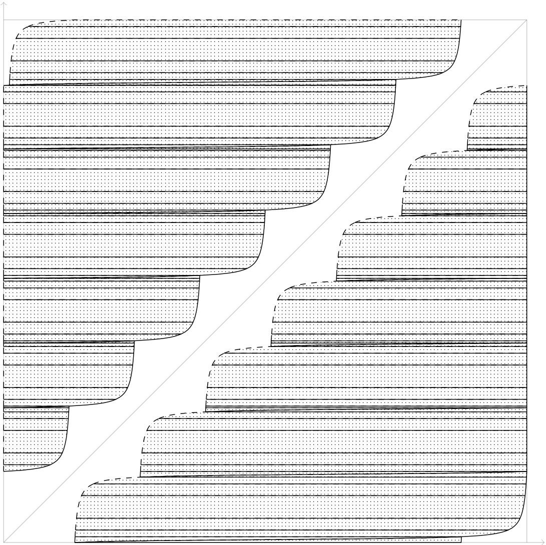
The special flow over B with roof function

$$\begin{array}{rccl} \tau : & B & \to & \mathbb{R}^+ \\ & (x,y) & \mapsto & \mathit{length}(\mathcal{D} \cap (x,y)) \end{array}$$

is conjugated with the geodesic flow on M.

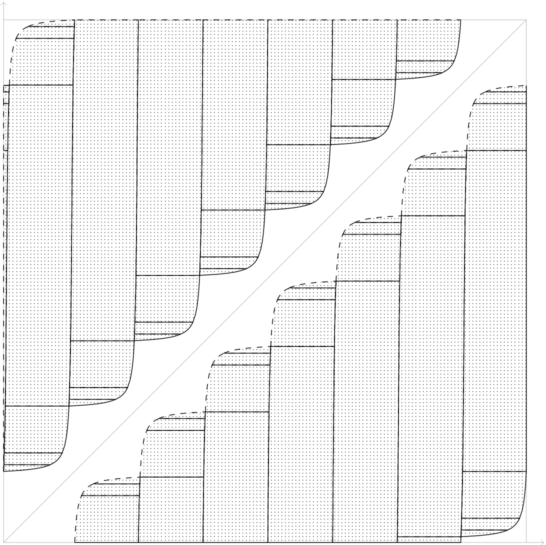
Geodesic billiard

Tubes of the billiard for the genus 2 surface

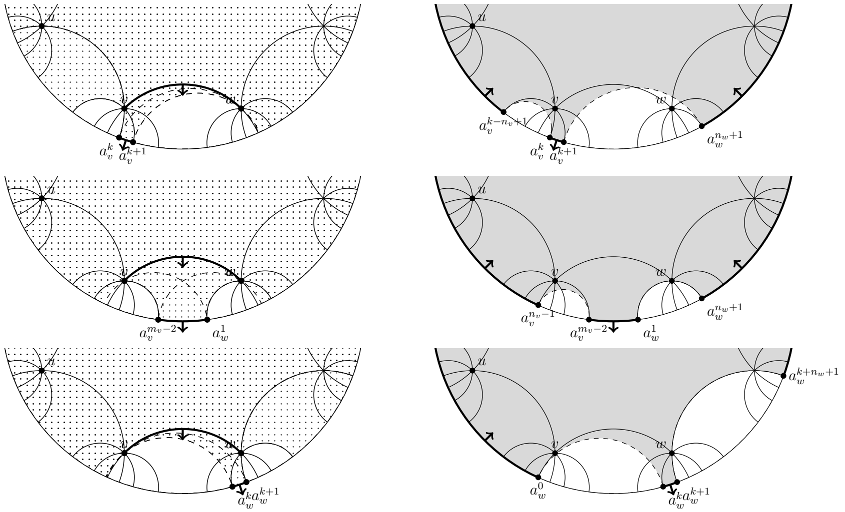


Geodesic billiard

Image of the billiard by T_B



Coding



u
v
w
a_v^k
a_v^{k+1}
$a_v^{k-n_v+1}$
$a_w^{n_w+1}$
$a_v^{m_v-2}$
a_w^1
$a_v^{n_v-1}$
a_w^k
a_w^{k+1}
$a_w^{k+n_w+1}$
a_w^0

Coding

Definition

For $v \in V$ and $w = r(v)$, let $C_v^{-n_v+1}, \ldots, C_v^0, \ldots, C_v^{n_w-2}$ be :

$$\forall k \in [\![n_v - 1 \,;\, m_v - 3]\!], C_v^{k-m_v+2} = \left] a_w^{n_w+1} ; a_v^{k-n_v+1} \right] \times \left[a_v^k ; a_v^{k+1} \right[$$

$$C_v^0 = \left] a_w^{n_w+1} ; a_v^{n_v-1} \right] \times \left[a_v^{m_v-2} ; a_w^1 \right[$$

$$\forall k \in [\![1 \,;\, n_w - 2]\!], C_v^k = \left] a_w^{k+n_w+1} ; a_w^0 \right] \times \left[a_w^k ; a_w^{k+1} \right[$$

Definition

The *right coding* is defined by :

$$C = \bigsqcup_{v \in V} \bigsqcup_{k=-n_v+1}^{n_{r(v)}-2} C_v^k$$

$$\forall x \in C_v^k, T_C(x) = \gamma_v(x)$$

Theorem

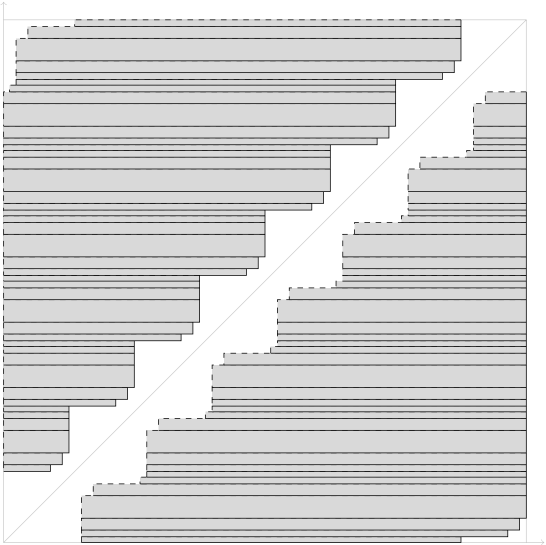
$T_C : C \to C$ is a bijection of C such that

$$T_C(x,y) = (S_L(x,y), T_R(y)) = (\gamma_R[y](x), \gamma_R[y](y))$$
$$T_C^{-1}(x,y) = (T_L(x), S_R(x,y)) = (\gamma_L[x](x), \gamma_L[x](y))$$

where $\gamma_L[x], \gamma_R[y] \in \Gamma$.
Moreover, $T_R : \mathbb{S}^1 \to \mathbb{S}^1$ (and T_L) preserves a finite Markov partition ($I_v^k = \pi_2(C_v^k)$).

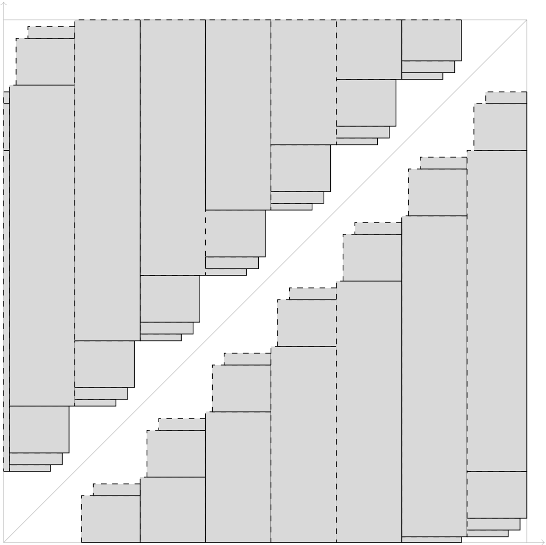
Coding

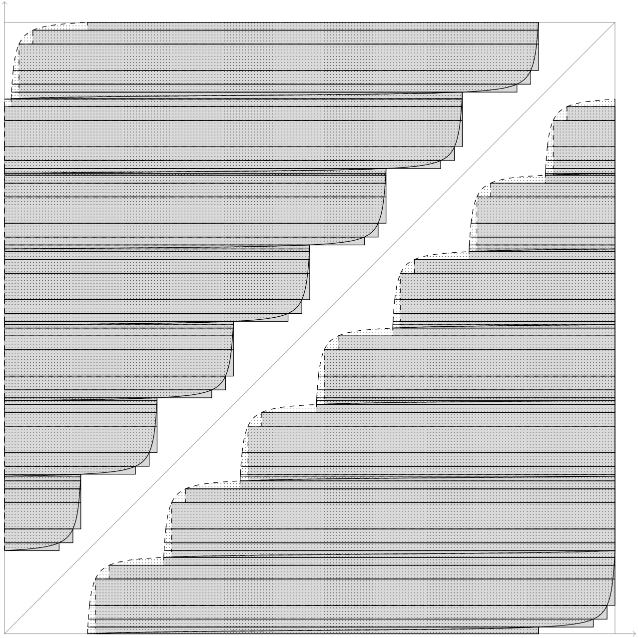
Rectangles of the right coding for the genus 2 surface



Coding

Image of the right coding by T_C





Conjugacy

Specific parts

$$B \cup C = (B \cap C) \sqcup \underbrace{\bigsqcup_{v \in V} \bigsqcup_{k=0}^{n_v - 1} X_v^k}_{=B \setminus C} \sqcup \underbrace{\bigsqcup_{v \in V} \bigsqcup_{k=0}^{n_v - 1} Y_v^{-k}}_{=C \setminus B}$$

Conjugacy

Specific parts

$$B \cup C = (B \cap C) \sqcup \underbrace{\bigsqcup_{v \in V} \bigsqcup_{k=0}^{n_v - 1} X_v^k}_{= B \setminus C} \sqcup \underbrace{\bigsqcup_{v \in V} \bigsqcup_{k=0}^{n_v - 1} Y_v^{-k}}_{= C \setminus B}$$

$$B \cap C = \underbrace{\Sigma}_{T_{B/\Sigma} = T_{C/\Sigma}} \sqcup \bigsqcup_{v \in V} L_v$$

Conjugacy

Specific parts

$$B \cup C = (B \cap C) \sqcup \underbrace{\bigsqcup_{v \in V} \bigsqcup_{k=0}^{n_v - 1} X_v^k}_{= B \setminus C} \sqcup \underbrace{\bigsqcup_{v \in V} \bigsqcup_{k=0}^{n_v - 1} Y_v^{-k}}_{= C \setminus B}$$

$$B \cap C = \underbrace{\Sigma}_{T_{B/\Sigma} = T_{C/\Sigma}} \sqcup \bigsqcup_{v \in V} L_v$$

$$\begin{array}{ccc} \Sigma & \xrightarrow{T_B} & \Sigma' \\ \downarrow \text{id} & & \downarrow \text{id} \\ \Sigma & \xrightarrow{T_C} & \Sigma' \end{array} \qquad \text{where } \Sigma' = T_B(\Sigma) = T_C(\Sigma) \subset B \cap C$$

Conjugacy

Theorem (A)

Let $\varphi_v^p = \gamma_{I\tau^{p-1}(v)} \dots \gamma_{I\tau(v)}\gamma_{I(v)}$. *Then*

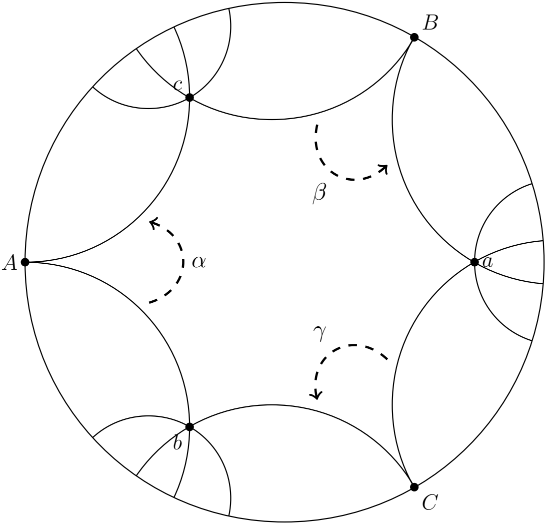
$$\begin{array}{lll} \varphi & : B & \to C \\ & x \in B \cap C & \mapsto x \\ & x \in X_v^k, k \in [\![1\,;\, n_v - 2]\!] & \mapsto \varphi_v^{2k+2}(x) \in Y_{\tau^{2k+2}(v)}^{-k} \\ & x \in X_v^0 \sqcup X_{I(v)}^{n_{I(v)}-1} & \mapsto \varphi_v^2(x) \in Y_{\tau^2(v)}^0 \sqcup Y_{r\tau^2(v)}^{-n_{r\tau^2(v)}+1} \end{array}$$

induces a bijection from B into C such that

$$\begin{array}{ccc} B & \xrightarrow{T_B} & B \\ \downarrow \varphi & & \downarrow \varphi \\ C & \xrightarrow{T_C} & C \end{array}$$

Example of a group of signature (0; 2; 3)

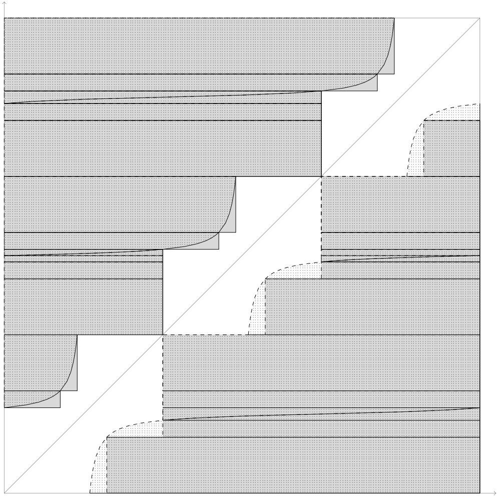
Fundamental domain



- A, B, C cusps
- a, b, c inner vertices
- $(\alpha\beta\gamma)^2 = \text{id}$
- $\sigma(a) = c$, $\sigma(c) = b$, $\sigma(b) = a$
- $\tau(a) = b$, $\tau(b) = c$, $\tau(c) = a$
- $n_v = 3$ for all v

Specific parts

Billiard and right codings together



Conjugacy

Complete expression

$$\begin{array}{llll}
\varphi : & B & \to C & \\
& x \in B \cap C & \mapsto x & \\
& x \in X_a^0 & \mapsto \varphi_a^2(x) = \alpha\gamma(x) & \in Y_c^0 \\
& x \in X_a^1 & \mapsto \varphi_a^4(x) = \gamma\beta\alpha\gamma(x) & \in Y_b^{-1} \\
& x \in X_b^0 & \mapsto \varphi_b^2(x) = \beta\alpha(x) & \in Y_a^0 \\
& x \in X_b^1 & \mapsto \varphi_b^4(x) = \alpha\gamma\beta\alpha(x) & \in Y_c^{-1} \\
& x \in X_c^0 & \mapsto \varphi_c^2(x) = \gamma\beta(x) & \in Y_b^0 \\
& x \in X_c^1 & \mapsto \varphi_c^4(x) = \beta\alpha\gamma\beta(x) & \in Y_a^{-1}
\end{array}$$

Conjugacy

Commutativity for the L_a tube

$$\begin{array}{ccc} L_a & \xrightarrow{\gamma} & X_b^{1,C} \\ \Big\downarrow \text{id} & & \Big\downarrow \alpha\gamma\beta\alpha \\ L_a & \xrightarrow{\beta^{-1}} & Y_c^{-1,C} \end{array}$$

since

$$(\alpha\gamma\beta\alpha)\gamma = \alpha(\gamma\beta\alpha\gamma\beta\alpha)(\beta\alpha)^{-1} = \beta^{-1}.$$

Conjugacy

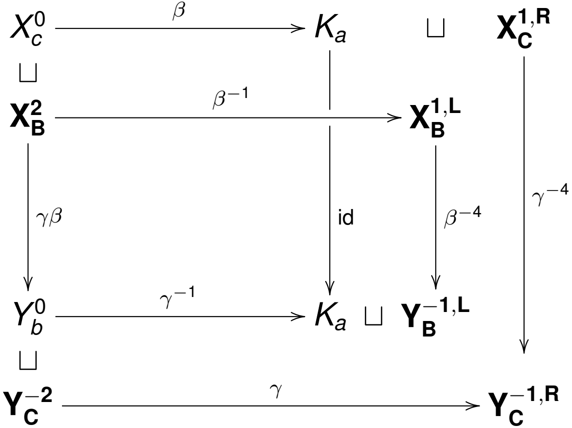
Commutativity for the X_b^1 tube

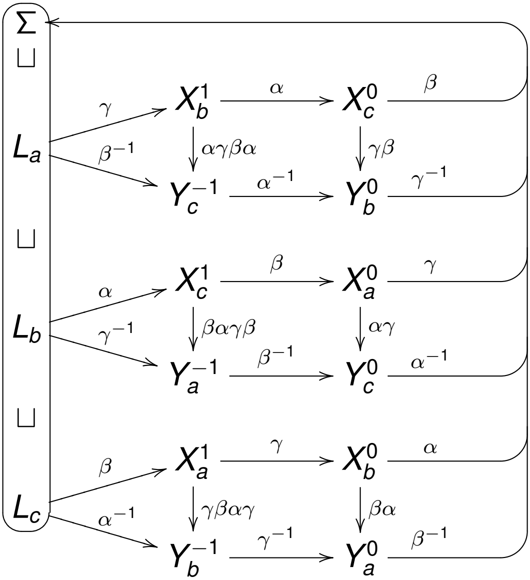
$$\begin{array}{ccc} X_b^1 & \xrightarrow{\alpha} & X_c^0 \sqcup \mathbf{X_B^2} \\ \downarrow{\scriptstyle \alpha\gamma\beta\alpha} & & \downarrow{\scriptstyle \gamma\beta} \\ Y_c^{-1} & \xrightarrow{\alpha^{-1}} & Y_b^0 \sqcup \mathbf{Y_C^{-2}} \end{array}$$

(bold sets are empty as they are based at a cusp)

Conjugacy

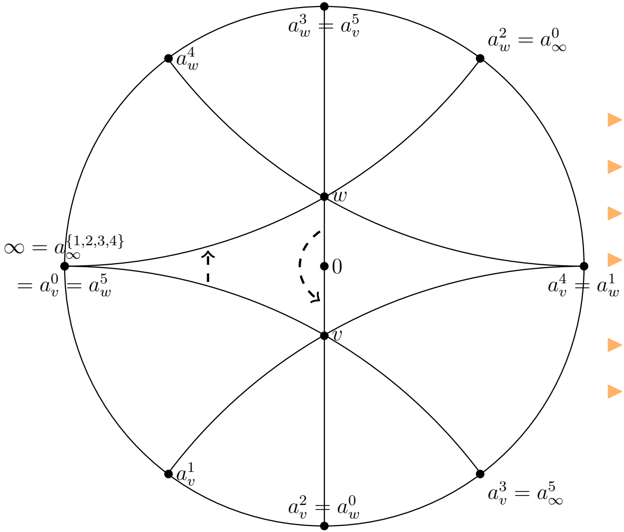
Commutativity for the X_c^0 tube




Σ
⊔
⊔
⊔
L_a
L_b
L_c
X_b^1
X_c^0
Y_c^{-1}
Y_b^0
X_c^1
X_a^0
Y_a^{-1}
Y_c^0
X_a^1
X_b^0
Y_b^{-1}
Y_a^0
γ
β^{-1}
α
γ^{-1}
β
α^{-1}
$\alpha\gamma\beta\alpha$
$\gamma\beta$
$\beta\alpha\gamma\beta$
$\alpha\gamma$
$\gamma\beta\alpha\gamma$
$\beta\alpha$

The example of the modular surface

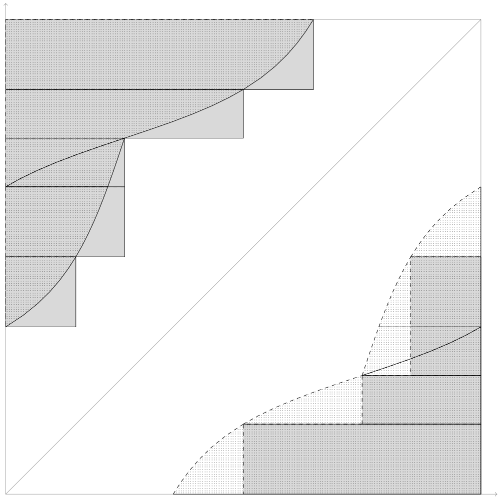
Fundamental domain



- ∞ cusp
- v, w inner vertices
- $S^2 = \text{id}$, $(ST)^3 = \text{id}$
- $\sigma(\infty) = \infty$, $\sigma(v) = w$, $\sigma(w) = v$
- $\tau = \sigma$
- $n_v = 3$ for all v

Specific parts

Billiard and right codings together



Conjugacy

Complete expression

$$
\begin{array}{rllll}
\varphi : & B & \to C & & \\
& x \in B \cap C & \mapsto x & & \\
& x \in X_v^0 & \mapsto \varphi_v^2(x) = ST(x) & \in Y_v^0 \sqcup Y_w^{-2} \\
& x \in X_v^1 & \mapsto \varphi_v^4(x) = T^{-1}S(x) & \in Y_v^{-1} \\
& x \in X_w^0 \sqcup X_v^2 & \mapsto \varphi_w^2(x) = TS(x) & \in Y_w^0 \\
& x \in X_w^1 & \mapsto \varphi_w^4(x) = ST^{-1}(x) & \in Y_w^{-1}
\end{array}
$$

Conjugacy

Commutativity for the L_v tube

$$
\begin{array}{ccc}
L_v & \xrightarrow{\;T\;} & X_w^{1,C} \\
\Big\downarrow \text{id} & & \Big\downarrow ST^{-1} \\
L_v & \xrightarrow{\;S\;} & Y_w^{-1,C}
\end{array}
$$

Conjugacy

Commutativity for the X_w^1 tube

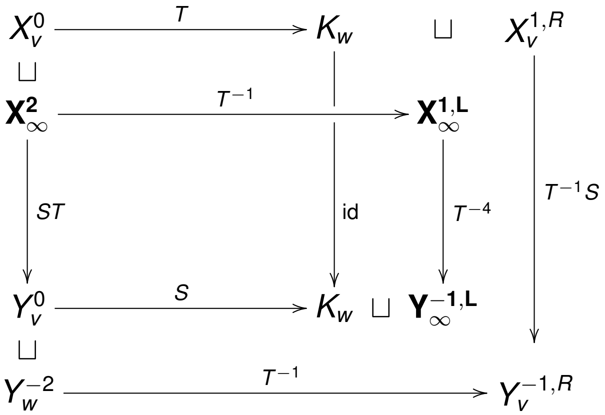
$$\begin{array}{ccc} X_w^1 & \xrightarrow{S} & X_v^0 \sqcup \mathbf{X}_\infty^{\mathbf{2}} \\ \downarrow{\scriptstyle ST^{-1}} & & \downarrow{\scriptstyle ST} \\ Y_w^{-1} & \xrightarrow{T^{-1}} & Y_v^0 \sqcup Y_w^{-2} \end{array}$$

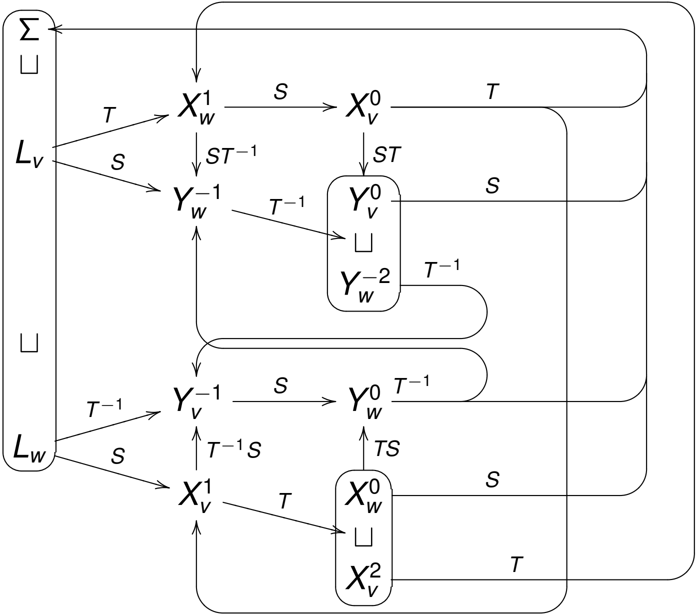
since

$$(ST)S = (STS)(TST)(TST)^{-1} = T^{-1}ST^{-1}.$$

Conjugacy

Commutativity for the X_v^0 tube




Σ
⊔
L_v
⊔
L_w
T
S
T^{-1}
S
X_w^1
X_v^0
Y_w^{-1}
Y_v^0
Y_w^{-2}
Y_v^{-1}
Y_w^0
X_v^1
X_w^0
X_v^2
ST^{-1}
ST
T^{-1}S
TS

Orbit-equivalence property and transfer operator

Orbit-equivalence

$T_C(x,y) = (S_L(x,y), T_R(y))$ where $T_R : \mathbb{S}^1 \to \mathbb{S}^1$ preserves a Markov partition $(I_v^k)_{v,k}$, and $T_{R/I_v} = \gamma_v \in \Gamma$.

Orbit-equivalence

$T_C(x,y) = (S_L(x,y), T_R(y))$ where $T_R : \mathbb{S}^1 \to \mathbb{S}^1$ preserves a Markov partition $(I_v^k)_{v,k}$, and $T_{R/I_v} = \gamma_v \in \Gamma$.

Definition

$T : \mathbb{S}^1 \to \mathbb{S}^1$ is said to be *orbit-equivalent* with the group Γ when :

$$\forall x, y \in \mathbb{S}^1, (\exists \gamma \in \Gamma, y = \gamma(x)) \Leftrightarrow (\exists p, q \geq 0, T^p(x) = T^q(y)).$$

Orbit-equivalence

$T_C(x,y) = (S_L(x,y), T_R(y))$ where $T_R : \mathbb{S}^1 \to \mathbb{S}^1$ preserves a Markov partition $(I_v^k)_{v,k}$, and $T_{R/I_v} = \gamma_v \in \Gamma$.

Definition

$T : \mathbb{S}^1 \to \mathbb{S}^1$ is said to be *orbit-equivalent* with the group Γ when :

$$\forall x, y \in \mathbb{S}^1, (\exists \gamma \in \Gamma, y = \gamma(x)) \Leftrightarrow (\exists p, q \geq 0, T^p(x) = T^q(y)).$$

Theorem (Series)

T_R is orbit-equivalent with Γ.

Invariance theorem

- $\mathbb{I} = \left\{ [a; b[\mid a, b \in \mathbb{S}^1 \right\}$.
- Γ acts on the left on $\mathbb{I}$ with $\gamma([a; b[) = [\gamma(a); \gamma(b)[$.
- Let X be another set on which Γ acts on the left. We say that $F : \mathbb{I} \times X \to \mathbb{C}$ *satisfies property* $\mathcal{I}(I, \gamma)$ when

$$\forall x \in X, F(I, x) = F(\gamma(I), \gamma(x)).$$

Invariance theorem

Theorem

Let X be a set on which Γ acts on the left, and suppose that $F : \mathbb{I} \times X \to \mathbb{C}$ satisfies the following properties :

Invariance theorem

Theorem

Let X be a set on which Γ acts on the left, and suppose that $F : \mathbb{I} \times X \to \mathbb{C}$ satisfies the following properties :

(i) (additivity for contiguous intervals) If $I, J \in \mathbb{I}$ are such that $I \sqcup J \in \mathbb{I}$, then :

$$\forall x \in X, F(I \sqcup J, x) = F(I, x) + F(J, x).$$

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Then $\mathcal{I}(I, \gamma)$ is true for every $I \in \mathbb{I}$ and $\gamma \in \Gamma$.

Application to orbit-equivalence

Fix $y \in \mathbb{S}^1$, and let :

- $X = \mathbb{S}^1$ with the usual left action of Γ.
- $F(I, x) = \begin{cases} 1 & \text{if } x \in I \text{ and : } \exists p, q \geq 0, T^p(x) = T^q(y) \\ 0 & \text{otherwise} \end{cases}$.

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This choice of X and F satisfies the hypothesis of the invariance theorem, hence :

$$\forall y, x \in \mathbb{S}^1, \qquad (\exists p, q \geq 0,\, T^p(x) = T^q(y)) \\ \Leftrightarrow (\forall \gamma \in \Gamma, \exists p, q \geq 0,\, T^p(\gamma(x)) = T^q(y))$$

This gives the orbit-equivalence theorem by picking $x = y$, in which case the first condition holds for $p = q = 0$.

Application to orbit-equivalence

Strong orbit equivalence

Theorem

$$\forall x \in \mathbb{S}^1, \forall \gamma \in \Gamma, \exists p, q \geq 0, \gamma_R^p[x] = \gamma_R^q[\gamma(x)]\gamma$$

where $\gamma_R^n[x] = T^n_{R/\{x\}}$.

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Pick :

- $X = \mathbb{S}^1 \times \Gamma$ with the action $\gamma(x, g) = (\gamma(x), g\gamma^{-1})$.
- $F(I, x) = \begin{cases} 1 & \text{if } x \in I \text{ and : } \exists p, q \geq 0, \gamma_R^p[x] = \gamma_R^q[g(x)]g \\ 0 & \text{otherwise} \end{cases}.$

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The invariance theorem shows then that :

$$\forall x \in \mathbb{S}^1, \forall g \in \Gamma, \quad (\exists p, q \geq 0, \gamma_R^p[x] = \gamma_R^q[g(x)]g)$$
$$\Leftrightarrow (\forall \gamma \in \Gamma, \exists p, q \geq 0, \gamma_R^p[\gamma(x)] = \gamma_R^q[\mathbf{g}(\mathbf{x})]g\gamma^{-1})$$

which gives the strong orbit-equivalence when $g = \text{id}$.

Transfer operator

Definition

Let E be the space of functions on $\mathbb{S}^1$ that are $\mathcal{C}^1$ on each $\overline{I_v^k}$.
We define the *transfer operator of T with parameter $s \in \mathbb{C}$* by

$$\begin{array}{rccc} \mathcal{L}_{T,s} : & E & \rightarrow & E \\ & f & \mapsto & \left(y \in \mathbb{S}^1 \mapsto \sum_{T(x)=y} \frac{f(x)}{|T'(x)|^s} \right) \end{array}.$$

Goal : finding eigendistributions of this operator.

Helgason boundary values

Theorem (Helgason)

Let $f \in \mathcal{C}^2(\mathbb{D})$ such that $\Delta f = -s(1-s)f$ and f grows at most exponentially in the hyperbolic radius. Then there exists an unique analytic distribution $\mathcal{D}_{f,s}$ on $\mathbb{S}^1$ such that

$$\forall z \in \mathbb{D}, f(z) = \langle \mathcal{D}_{f,s}, P^s(z,.) \rangle$$

where $P^s(z,x)$ is the Poisson kernel. More precisely, the mapping $T \mapsto (z \mapsto \langle T, P^s(z,.)\rangle)$ is an isomorphism from analytic distributions of $\mathbb{S}^1$ onto smooth eigenfunctions of the hyperbolic laplacian for the eigenvalue $-s(1-s)$.

Helgason boundary values

Invariance

Proposition

$$\forall \gamma \in \Gamma, \forall \varphi \in \mathcal{C}^\infty(\mathbb{S}^1), \left\langle \mathcal{D}_{f,s}, \frac{\varphi \circ \gamma^{-1}}{|\gamma' \circ \gamma^{-1}|^s} \right\rangle = \langle \mathcal{D}_{f\circ\gamma,s}, \varphi \rangle.$$

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Problem : $\mathcal{D}_{f,s}$ acts on $\mathcal{C}^\infty(\mathbb{S}^1)$ but not on E.

Helgason boundary values

Otal's approach

When $\alpha > 0$, let Λ_α be the space of α-Hölder contiunous functions on $[0; 2\pi[$ mapping 0 to 0.

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Definition (Weak derivative)

If f is a continuous function over $[0; 2\pi[$,

$$\langle f', \varphi \rangle = f(2\pi)\varphi(2\pi) - f(0)\varphi(0) - \int_0^{2\pi} \frac{\partial \varphi}{\partial x}(x) f(x) dx$$

for $\varphi \in \mathcal{C}^1([0; 2\pi[)$.

Helgason boundary values

Extension to E

Theorem (Otal)

Let $f \in \mathcal{C}^2(\mathbb{D})$ such that $\Delta f = -s(1-s)f$ and f grows at most exponentially in the hyperbolic radius. Then $\mathcal{D}_{f,s}$ is the weak derivative of a function $D_{f,s} \in \Lambda_{\Re(s)}$.

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If $I = [a; b[$, we can now extend $\mathcal{D}_{f,s}$ to $\mathcal{C}^1(\bar{I})$ with

$$\forall \varphi \in \mathcal{C}^1(\bar{I}), \langle \mathcal{D}_{f,s}, \varphi \mathbb{1}_I \rangle = \varphi(b) D_{f,s}(b) - \varphi(a) D_{f,s}(a) - \int_a^b \frac{\partial \varphi}{\partial x}(x) D_{f,s}(x) dx.$$

Hence $\mathcal{D}_{f,s}$ can be extended to E.

Helgason boundary values

$\mathcal{D}_{f,s}$ is an eigendistribution of $\mathcal{L}_{T_R,s}$

Proposition

$$\forall \gamma \in \Gamma, \forall \varphi \in \mathcal{C}^1(\bar{I}), \left\langle \mathcal{D}_{f,s}, \frac{\varphi \circ \gamma^{-1}}{|\gamma' \circ \gamma^{-1}|^s} \mathbb{1}_{\gamma(I)} \right\rangle = \langle \mathcal{D}_{f\circ\gamma,s}, \varphi \mathbb{1}_I \rangle.$$

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Theorem (Lopes-Thieullen)

Let f be such that $\Delta f = -s(1-s)f$, $f \circ \gamma = f \ \forall \gamma \in \Gamma$. Then :

$$\forall \varphi \in E, \langle \mathcal{D}_{f,s}, \mathcal{L}_{T_R,s}\varphi \rangle = \langle \mathcal{D}_{f,s}, \varphi \rangle.$$

Extension of Pollicott's theorem

Theorem (B)

Let ν be a bounded operator on E that is the weak derivative of a function of Λ_σ with $0 < \sigma \leq 1$. Note $s = \sigma + i\omega$ for some $\omega \in \mathbb{R}$. Then

$$\forall \varphi \in E, \langle \nu, \mathcal{L}_{T_R,s}\varphi \rangle = \langle \nu, \varphi \rangle$$

if and only if

$$\begin{array}{rccc} f_\nu & : & \mathbb{D} & \rightarrow & \mathbb{C} \\ & & z & \mapsto & \langle \nu, P^s(z,.) \rangle \end{array}$$

satisfies $\Delta f_\nu = -s(1-s) f_\nu$ and $f_\nu \circ \gamma = f_\nu$ for every $\gamma \in \Gamma$, i.e. if and only if ν is the Helgason boundary value of an eigenfunction of the laplacian on M.

Extension of Pollicott's theorem

Sketch of the proof

- We use the invariance theorem for $X = \mathbb{D}$ and

$$F(I, x) = \langle \nu, P^s(x, .)\mathbb{1}_I(.)\rangle.$$

- The inclusion property is verified because ν is the weak derivative of a continuous function.

Extensions

- Geometrically finite Fuchsian groups ?

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- Eigenfunctions of the transfer operator ?

Thanks

Thank you for your attention.