Ruelle operator duality
for smooth coupled Markov maps of the circle

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This document presents how two expanding Markov maps coupled by Baker-like transforms can present an explicit duality between eigenelements for the Ruelle operator of each map. This setting has been inspired by the left and right Bowen-Series maps associated with even corners fundamental domains for hyperbolic surfaces of finite volume.

1. Coupled Markov maps of the circle

In all the following, T_L (respectively T_R) will be a surjective expanding map of the circle to itself that preserves a finite partition $\mathcal{I}_L = (I^L_k)_{k=1\dots N}$ (respectively $\mathcal{I}_R = (I^R_k)_{k=1\dots N}$) of the circle in left-open (respectively right-open) intervals. Moreover, each branch T_{L/I^L_k} (respectively T_{L/I^L_k}) will be assumed to be a $\mathcal{C}^1$-diffeomorphism onto its image.

Definition 1.1. We say that two such maps are *coupled* when there is a subset C of $\mathbb{S}^1 \times \mathbb{S}^1$ and a map $T_C : C \to C$ such that :

1. C is a connex union of squares of the form $I^L_j \times I^R_k$: there are two collections of intervals $(J^L_k)_{k=1\dots N}$ and $(J^R_k)_{k=1\dots N}$, each of those being a union of contiguous I^L_j and I^R_j respectively, and such that :
$$C = \bigsqcup_{j=1}^{N} I^L_j \times J^R_j = \bigsqcup_{k=1}^{N} J^L_k \times I^R_k$$
2. T_C is a bijection of C to itself, is a skew-product of base T_L, and its inverse is a skew-product of base T_R : there are two collections of maps $(S_R(j,.))_{j=1\dots N}$ and $(S_L(.,k))_{k=1\dots N}$ such that for every $(x,y) \in C$:
$$T_C(x,y) = (T_L(x), S_R(\kappa_L(x), y))$$
$$T_C^{-1}(x,y) = (S_L(x, \kappa_R(y)), T_R(y))$$
where $\kappa_L(x)$ (respectively $\kappa_R(y)$ is the interval of the partition $\mathcal{I}_L$ (respectively $\mathcal{I}_R$) in which x (respectively y) lies.

Some examples :

- If $T_L(z) = T_R(z) = z^2$ (i.e. the usual doubling map), T_L and T_R are coupled by the Baker map defined over $C = \mathbb{S}^1 \times \mathbb{S}^1$.
- Suppose that T_L is a Markov map relatively to the partition $\mathcal{I} = (I_k)_{k=1\dots N}$ whose transition matrix M^L is such that its transpose $M^R = {}^tM^L$ is still the transition matrix of a Markov map. Take T_R any Markov map relatively to $\mathcal{I}$ that has for transitions M^R. Then they are coupled by the given of :
$$C = \bigsqcup_{k=1}^{N} I_k \times T_R(I_k) = \bigsqcup_{k=1}^{N} T_L(I_k) \times I_k$$
$$T_C(x,y) = (T_L(x), T^{-1}_{R/I_{\kappa(x)}}(y))$$
$$T_C^{-1}(x,y) = (T^{-1}_{L/I_{\kappa(y)}}(x), T_R(y))$$
- The left and right Bowen-Series maps for cofinite Fuchsian groups are coupled (see below).

We will also use the following notations :

$$J_L(y) = C \cap \mathbb{S}^1 \times \{y\} \qquad J_R(x) = C \cap \{x\} \times \mathbb{S}^1$$
$$S^n_R(x,y) = \pi_2(T^n_C(x,y)) \qquad S_R(x,y) = S^1_R(x,y) = S_R(\kappa_L(x), y))$$
$$S^n_L(x,y) = \pi_1(T^{-n}_C(x,y)) \qquad S_L(x,y) = S^1_L(x,y) = S_L(x, \kappa_R(y))$$

2. Involution kernel between two potentials

Let $A_L : \mathbb{S}^1 \to \mathbb{C}$ and $A_R : \mathbb{S}^1 \to \mathbb{C}$ be two potentials respectively associated with T_L and T_R. We will assume a non-symmetric set of hypothesis on those potentials :

- A_L will be supposed to have bounded variations.
- A_R will be supposed to be absolutely continuous on the closure of each interval of the partition $\mathcal{I}_R$.

The Ruelle operators associated with the systems (T_L, A_L) and (T_R, A_R) are defined by :

$$\mathcal{L}_L f(x') = \sum_{T_L(x)=x'} e^{A_L(x)} f(x)$$
$$\mathcal{L}_R f(y') = \sum_{T_R(y)=y'} e^{A_R(y)} f(y)$$

They act respectively on the space of functions with bounded variations and the space of piecewise $\mathcal{C}^1$ maps according to the partition $\mathcal{I}_R$.

Definition 2.1. Two potentials A_L and A_R associated with Markov maps T_L and T_R coupled by an extension (C, T_C) are *in involution* when there is a map $W : C \to \mathbb{C}$ such that :

$$\forall (x,y) \in C, A_L(x) + W(x,y) = W(T_C(x,y)) + A_R(S_R(x,y))$$

If T_L, T_R are uniformly expanding and A_L is Hölder, then there always exists an Hölder potential A_R which is involution with A_L by an Hölder kernel W. Moreover, the difference between two suitable kernels for A_L and A_R may only depend on y. In this setting, the classical technique for building such kernels is to use Sinaï's method. For $x, x', y \in \mathbb{S}^1$ such that $(x,y) \in C$ and $(x',y) \in C$, let :

$$\Delta(x,x',y) = \sum_{n \geq 1} A_L(S^n_L(x,y)) - A_L(S^n_L(x',y))$$

Take μ a measure such that $\mathcal{L}^*_L \mu = \lambda_{\max} \mu$ with $\lambda_{\max}$ the leading eigenvalue. Then :

$$\Delta(x,x',y) - \log \int_{J_L(y)} \exp(\Delta(u,x',y)) d\mu(u)$$

does not depend on x' and is an involution kernel for A_L.

We will assume onwards that the potentials A_L and A_R are in involution by a kernel W such that $y \mapsto W(x,y)$ is absolutely continuous on the fiber $J_R(x)$ for every x.

3. Eigendistributions of a Ruelle operator acting on a space of piecewise smooth functions

Since $\mathbb{S}^1$ is compact, there cannot be any distribution of infinite order in the classical sense. Hence a *distribution over* $\mathbb{S}^1$ is just a continuous linear functional over one of the spaces $\mathcal{C}^k(\mathbb{S}^1)$ equipped with its usual norm. The lowest suitable index k is the *order* of the distribution. We will find the distributions we want to study among those of order 1 :

Definition 3.1. A distribution ν over $\mathbb{S}^1$ is the *weak derivative of a continuous map* if there is a map $h : \mathbb{R} \to \mathbb{C}$ continuous such that $h(x + 2\pi) = h(x) + c$ for some c and for which :

$$\forall \varphi \in \mathcal{C}^1(\mathbb{S}^1), \langle \nu, \varphi \rangle = \tilde{\varphi}(2\pi) h(2\pi) - \tilde{\varphi}(0) h(0) - \int_0^{2\pi} \tilde{\varphi}'(t) h(t) dt$$

where $\tilde{\varphi}$ is the lift of φ to $\mathbb{R}$. We will note $\nu = h'$.

Thanks to this expression, these distributions can be extended to act on functions φ that are $\mathcal{C}^1$ only over an interval $I = [a;b[$ of $\mathbb{S}^1$ by letting :

$$\langle \hat{\nu}, \varphi \mathbb{1}_{[a;b[} \rangle = \tilde{\varphi}(\tilde{b}) h(\tilde{b}) - \tilde{\varphi}(\tilde{a}) h(\tilde{a}) - \int_{\tilde{a}}^{\tilde{b}} \tilde{\varphi}'(t) h(t) dt$$

where $\left[\tilde{a}; \tilde{b}\right[$ is a lift of $[a;b[$ to $\mathbb{R}$. By additivity, this extension can take for test functions the piecewise $\mathcal{C}^1$ functions onto which the Ruelle operator $\mathcal{L}_R$ acts. This allows us to talk about eigendistributions of $\mathcal{L}_R$.

Definition 3.2. A distribution $\nu = h'$ is an *eigendistribution of* $\mathcal{L}_R$ if there is an eigenvalue $\lambda \in \mathbb{C}$ such that for every function φ that is $\mathcal{C}^1$ on each interval of the partition $\mathcal{I}_R$ we have :

$$\langle \hat{\nu}, \mathcal{L}_R \varphi \rangle = \lambda \langle \hat{\nu}, \varphi \rangle$$

In the following, we will just note ν for the extension $\hat{\nu}$.

4. Ruelle operator duality

The duality between eigendistributions of $\mathcal{L}_R$ and the eigenfunctions of $\mathcal{L}_L$ is formally established by the map :

$$\Phi : \nu \to \left(x \mapsto \langle \nu, e^W(x,.) \mathbb{1}_C(x,.) \rangle \right)$$

The injectivity of this map is the easiest to achieve :

Theorem 4.1 (Injectivity of Φ). *Let $\nu = h'$ be a non-zero eigendistribution of $\mathcal{L}_R$ for the eigenvalue λ. Then $\psi = \Phi(\nu)$ is a non-zero eigenfunction of $\mathcal{L}_L$ for the same eigenvalue λ.*

The surjectivity requires more hypothesis to be verified, but is constructive :

Theorem 4.2 (Surjectivity of Φ). *Let ψ be an eigenfunction of $\mathcal{L}_L$ for the eigenvalue λ, and assume that :*

- *ψ has bounded variations.*
- *W is bounded from above and below.*
- *For every x, the function series :*
$$y \mapsto \sum_{n \geq 0} \lambda^{-n} e^{A^n_L(S^n_L(x,y))}$$
converges uniformly on the fiber $J_R(x)$.

Then there exists a eigendistribution $\nu = h'$ (with h continuous) of $\mathcal{L}_R$ for the eigenvalue λ such that $\psi = \Phi(\nu)$. Moreover, ν can be explicitly written as a limit of functions constructed from ψ.

The most striking point is that the map Φ is the same for all the eigenvalues that satisfy the hypothesis of these results.

The explicit construction of the eigendistribution ν of $\mathcal{L}_R$ from an eigenfunction ψ of $\mathcal{L}_L$ goes as follows. First, by changing A_L and A_R into $A_L - \log \lambda$ and $A_R - \log \lambda$ (which are still in involution by the same kernel W), we can assume that $\lambda = 1$. Now let for every $n \geq 0$ and $(x', y') \in C$:

$$g_n(x', y') = \sum_{T^n_L(x)=x'} e^{A^n_L(x)} \psi(x) \mathbb{1}_{[c(x');y']}(S^n_R(x, d(x)))$$

where the fiber $J_R(x) = [c(x); d(x)[$. Note that $g_n(x', c(x')) = 0$ whereas $g_n(x', d(x')) = \psi(x')$ for every n. Under the hypothesis of the theorem, this sequence of functions converges for every x' uniformly in $y' \in J_R(x')$ to a $g : C \to \mathbb{C}$ such that :

$$\forall (x', y') \in C, g(T_C(x', y')) = e^{A_L(x')} g(x', y') + \eta(x')$$

for some η bounded. Moreover, $y' \mapsto g(x', y')$ is continuous on $J_R(y')$ for every x' From there, define :

$$h(x', y') = e^{-W(x',y')} g(x', y') - e^{-W(x', c(x'))} g(x', c(x')) - \int_{c(x')}^{y'} \left[\partial_2 e^{-W}\right](x', t) g(x', t) dt$$

It is now possible to show that for every pair of intervals I^L_j, I^L_k that have intersecting fibers $J^R_j \cap J^R_k \neq \emptyset$, there is a constant $\mu_{j,k} \in \mathbb{C}$ such that :

$$\forall x'_1 \in I^L_j, x'_2 \in I^L_k, y' \in J^R_j \cap J^R_k, h(x'_1, y') - h(x'_2, y') = \mu_{j,k}$$

which means that $h(x', y')$ "almost" does not depend on x'. Finally, if $I^R_k = [y_k; y_{k+1}[$, we fix for every k a x_k such that $(x_k, y_k) \in C$, and we let :

$$h_1 = 0$$
$$\forall k, h_{k+1} = h_k + h(x_k, y_k) - h(x_{k+1}, y_k)$$
$$\forall k, \forall y \in I^R_k, h(y) = h_k + h(x_k, y)$$

This map h can be lifted to a continuous map on $\mathbb{R}$, and $\nu = h'$ is an eigendistribution of $\mathcal{L}_R$ for the eigenvalue 1 such that $\psi = \Phi(\nu)$.

5. The example of the Bowen-Series maps

The *Bowen-Series maps* T_L and T_R are a couple of surjective expanding Markov maps of the circle naturally associated with "good" fundamental domains for the action of a Fuchsian group Γ of finite covolume on the hyperbolic plane $\mathbb{H}$. Each of the branches of these maps are hyperbolic isometries. When Γ is actually cocompact, the Bowen-Series maps are even uniformly expanding.

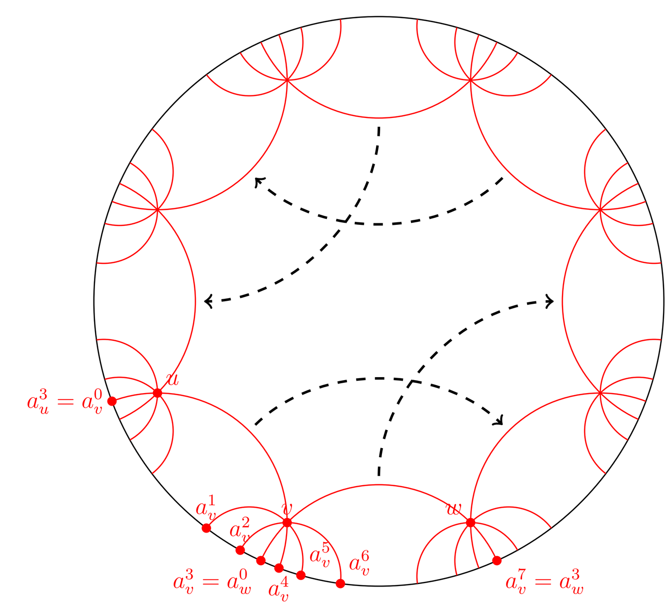

Figure 1: *A "good" fundamental domain for Σ_2*

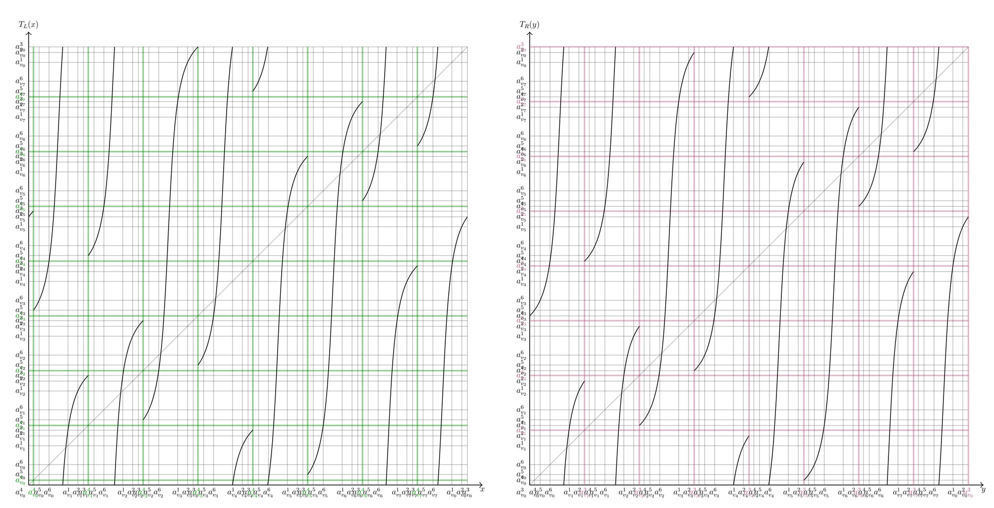

Figure 2: *The left and right Bowen-Series map for Σ_2*

They are always coupled by a Baker-like map T_C (which happens to be also conjugated with a Poincaré section of the geodesic flow on the surface) whose support C stays at bounded distance from the diagonal of $\mathbb{S}^1 \times \mathbb{S}^1$ when Γ is cocompact.

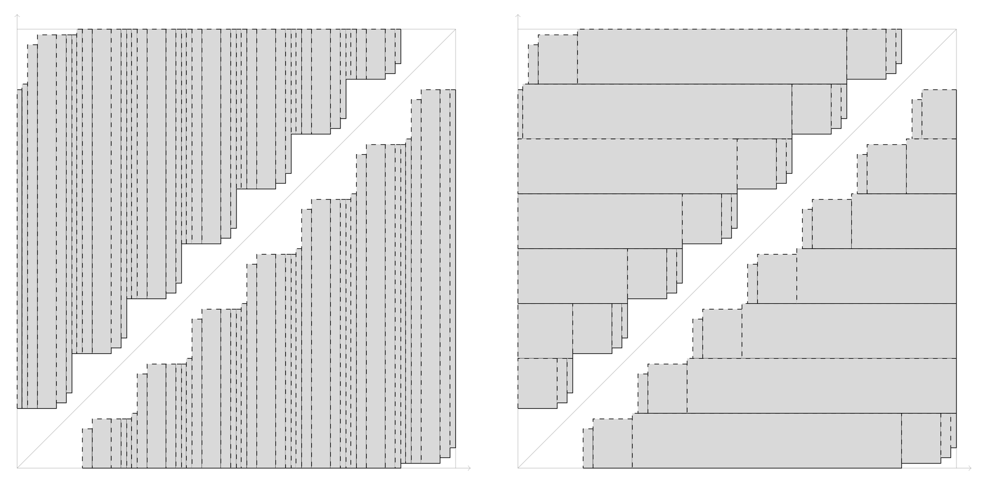

Figure 3: *The coupling Baker map for Σ_2*

Moreover, the potentials $A_L = -\log |T'_L|$ and $A_R = -\log |T'_R|$ are in involution by the kernel $W(x,y) = -\log |x - y|^2$ derived from the Gromov distance. Hence, whenever Γ is cocompact, both theorems can be applied to these maps equipped with these potentials. This gives an explicit relation between eigenfunctions of $\mathcal{L}_L$ for the eigenvalue 1 and eigendistributions of $\mathcal{L}_R$ for the eigenvalue 1, which are themselves explicitely related to eigenfunctions of the laplacian on the quotient surface $\mathbb{H}_{/\Gamma}$.

References

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