Linking geodesic flow and eigenfunctions of the laplacian on hyperbolic surfaces of finite volume

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This poster presents how the Bowen-Series transform is defined for surfaces of finite volume and how it can be related with both the geodesic flow and the eigenfunctions of the hyperbolic laplacian.

1. The Bowen-Series transform

We denote by $\mathbb{D}$ the unit complex disk equipped with the usual hyperbolic metric with constant curvature -1. The unit circle $\mathbb{S}^1$ is identified with its boundary at infinity $\partial\mathbb{D}$.
Let Γ be a discrete subgroup of hyperbolic isometries such that $\Gamma\backslash\mathbb{D}$ has finite hyperbolic volume. In particular, this implies that Γ is geometrically finite and that its limit set is is the whole unit circle $\mathbb{S}^1$.
Let $\mathcal{D}$ be a convex fundamental domain for the action of Γ on $\mathbb{D}$. Denote by V the set of its vertices (including vertices at infinity) and S the set of its sides ; both are finite. Any side $s \in S$ is paired with an unique $s' \in S$, and there exists $\gamma_s, \gamma_{s'} \in \Gamma$ such that $\gamma_s(s) = s'$ and $\gamma_s^{-1} = \gamma_{s'}$; and then $(\gamma_s)_{s\in S}$ is a generating set for Γ. If s is a side of $\mathcal{D}$, let $\tilde{s}$ be the complete geodesic supporting s.
The Bowen-Series transform can only be defined starting from very specific fundamental domains : they must satisfy the *even corners* property. $\mathcal{D}$ is said to be *even corners* when, for any side $s \in S$ and any $\gamma \in \Gamma$, the complete geodesic $\gamma(\tilde{s})$ does not meet the interior of $\mathcal{D}$. If Γ has no elliptic elements, it always has an even corners fundamental domain, but it may not be easily described by an algorithm (Dirichlet domain are almost surely non even corners).

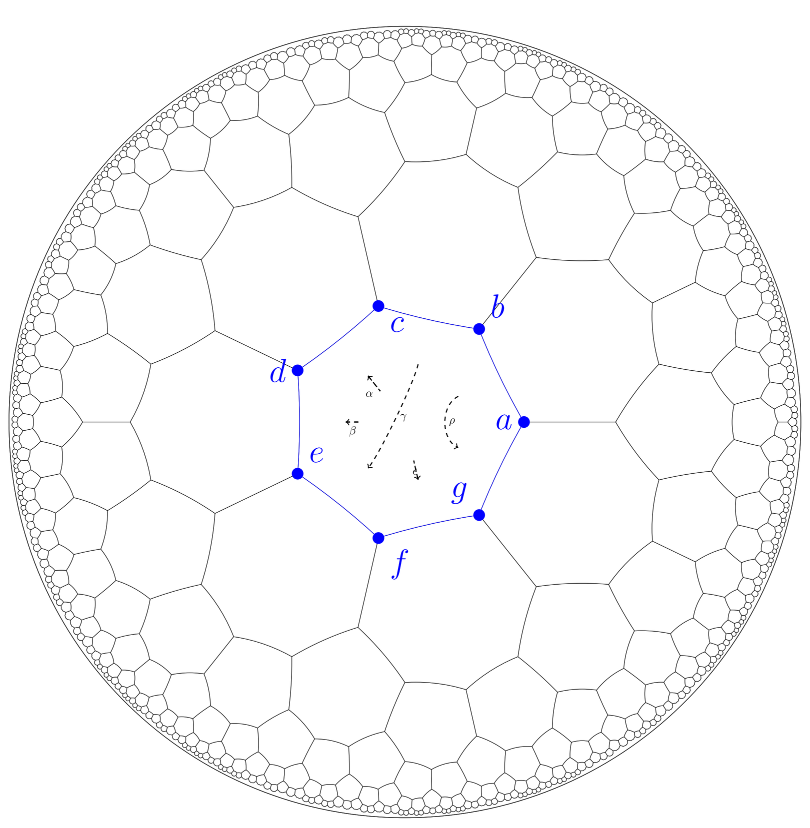

Figure 1: *A non even corners fundamental domain*

Let $\mathcal{N} = \{\gamma(\tilde{s}) \mid s \in S, \gamma \in \Gamma\}$. For any $v \in V$, the finite set $\mathcal{N}_v = \{g \in \mathcal{N} \mid v \in g\}$ marks a finite number of endpoints at infinity :
– when v is an inner vertex, $\mathcal{N}_v$ contains $m_v = 2n_v$ elements. We start by calling a_v^0 the furthest endpoint of the left side that goes through v, and likewise $a_v^{m_v-1}$ for the right side. By the even corners hypothesis, all other endpoints are in $\left]a_v^0; a_v^{m_v-1}\right[$; so we order them a_v^k for $0 < k < m_v - 1$.
– when v lies on the boundary, we arbitrarily force $n_v = 3$, so that $a_v^1 = a_v^2 = a_v^3 = a_v^4 = v$.

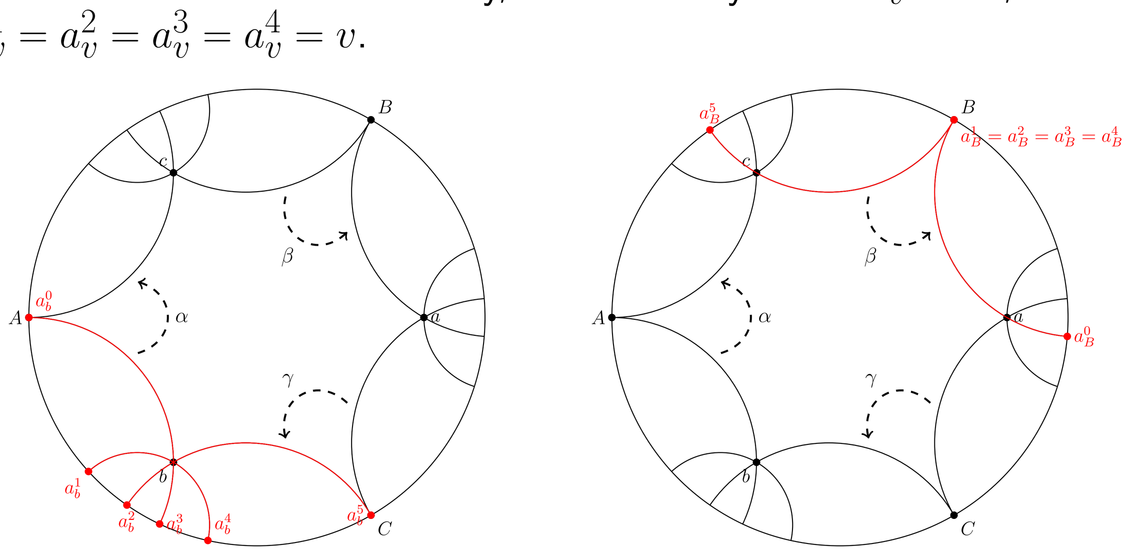

Figure 2: *Endpoints for an inner vertex (left) and a cusp (right)*

If $v \in V$, denote by w the vertex of $\mathcal{D}$ immediately following v in the trigonometric order. Let γ_v be the generator associated with the right side of the domain passing through v. $I_v^L = \left]a_v^{n_v}; a_w^{n_w}\right]$ and $I_v^R = \left[a_v^{n_v-1}; a_w^{n_w-1}\right[$ are two partitions of the $\mathbb{S}^1$ in intervals. We define the *left and right Bowen-Series transformations* by :

$$T_L : \begin{array}{rcl} \mathbb{S}^1 & \to & \mathbb{S}^1 \\ x \in I_v^L & \mapsto & \gamma_v(x) \end{array} \quad \text{and} \quad T_R : \begin{array}{rcl} \mathbb{S}^1 & \to & \mathbb{S}^1 \\ x \in I_v^R & \mapsto & \gamma_v(x) \end{array}$$

T_L and T_R preserve the partitions $(I_k^L)_k$ and $(I_k^R)_k$ of $\mathbb{S}^1$ in respectively left-open and right-open intervals delimited by the a_v^k. They also are expansive and transitive maps.

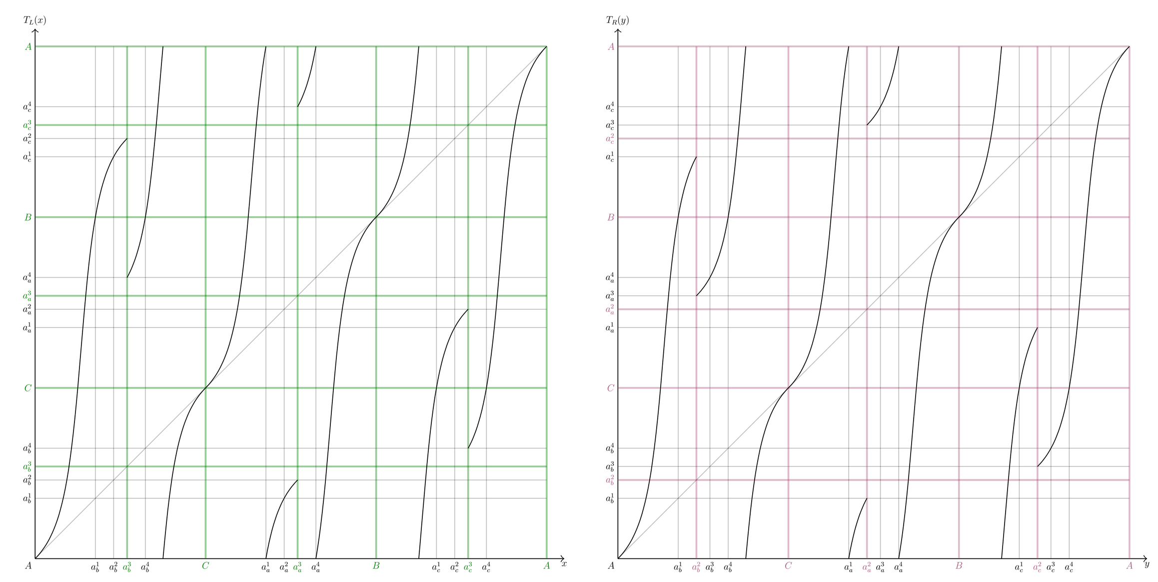

Figure 3: *T_L and T_R for the previous example*

The action of these transformations on $\mathbb{S}^1$ is well-known for being orbit-equivalent with the action of Γ at infinity :

Theorem 1.1 (Series, [1]). *Let $T = T_L$ or T_R. For every $x, y \in \mathbb{S}^1$,*

$$\exists \gamma \in \Gamma, y = \gamma(x) \Leftrightarrow \exists p, q \geq 0, T^p(x) = T^q(y)$$

This result is the expression of a more general *invariance principle*. Note $\mathbb{I} = \left\{\left[e^{ia}; e^{ib}\right[\mid 0 \leq a \leq b \leq 2\pi\right\}$. Let Z be a set on which Γ acts on the left. Γ acts also naturally on the left on $\mathbb{I}$.
We say that $F : \mathbb{I} \times Z \to \mathbb{C}$ satisfies $\mathcal{I}(I, \gamma)$ when :

$$\forall z \in Z, F(I, z) = F(\gamma(I), \gamma(z)).$$

Theorem 1.2. *Let $F : \mathbb{I} \times Z \to \mathbb{C}$ such that :*
(i) (Additivity for contiguous intervals) If $I, J \in \mathbb{I}$ are contiguous and disjoint, then $F(I \sqcup J, z) = F(I, z) + F(J, z)$ for every $z \in Z$;
(ii) (Continuity) If $(b_n)_{n\geq 0} \to b$, then $(F([a; b_n[, z))_{n\geq 0} \to F([a; b[, z)$ for every $z \in Z$;
(iii) (Inclusion) If F satisfies $\mathcal{I}(I, \gamma)$, then it satisfies $\mathcal{I}(J, \gamma)$ for every $J \subset I$, $J \in \mathbb{I}$;
(iv) (Bowen-Series invariance) F satisfies $\mathcal{I}(I_v^R, \gamma_v)$ for every $v \in V$.
Then F satisfies $\mathcal{I}(I, \gamma)$ for every $I \in \mathbb{I}$ and $\gamma \in \Gamma$.

Basically, it allows us to transport the combinatorics of intervals under the action of T_R to relations in Z. One can state a similar result for T_L as well. When applied for $Z = \mathbb{S}^1$ and :

$$F_y(x, I) = \begin{cases} 1 & \text{if } x \in I \text{ and } \exists p, q \geq 0, T^p(x) = T^q(y) \\ 0 & \text{otherwise} \end{cases}$$

it gives back Series' result. Applied with a suitable choice of F, this general result allows us to describe the periodic orbits of T_L and T_R :

Theorem 1.3. *There is a bijection between periodic hyperbolic orbits of T_L/T_R and conjugacy classes of primitive hyperbolic elements of Γ.*

In particular, this result leads to an expression of the dynamical zeta functions of T_L and T_R in terms of the Selberg zeta function of Γ without a correction term for boundary geodesics as in [4].

2. Natural extension and the geodesic flow on the quotient

We will now explain how the Bowen-Series transforms can be related with the geodesic flow on the quotient $\Gamma\backslash\mathbb{D}$.
We proved that T_L and T_R can respectively be seen as the first factor of a Baker-like transform and the right factor of its inverse :

Theorem 2.1. *There exists a set $C \subset \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and a bijection $T_C : C \to C$ such that :*
(i) C is a finite reunion of rectangles : there are intervals J_k^L, J_k^R such that :

$$C = \bigsqcup_{k=1}^{n} I_k^L \times J_k^R = \bigsqcup_{k=1}^{n} J_k^L \times I_k^R$$

(ii) For every $(x, y) \in C$:

$$T_C(x, y) = (\gamma_L[x](x), \gamma_L[x](y)) = (T_L(x), S_R(x, y))$$
$$T_C^{-1}(x, y) = (\gamma_R[y](x), \gamma_R[y](y)) = (S_L(x, y), T_R(y))$$

Any oriented geometric geodesic of $\Gamma\backslash\mathbb{D}$ can be lifted to $\mathbb{D}$ where it is parametrized by its endpoints $(x, y) \in \mathbb{T}^2 \setminus \Delta$. Among all the possible lifts, one and only one lies in the set B of all geodesics of $\mathbb{D}$ that either cross the interior of $\mathcal{D}$ or pass through a vertex $v \in V$ while keeping the fundamental domain on their right. We set $T_B(x, y) = (\gamma_s(x), \gamma_s(y))$ whenever $(x, y) \in B$ leaves the fundamental domain by the side s. T_B is a bijection of B, and (B, T_B) is called the *geodesic billiard* associated with $\Gamma\backslash\mathbb{D}$. The geodesic flow on $\Gamma\backslash\mathbb{D}$ can be obtained as the suspension of (B, T_B) by the transit time across $\mathcal{D}$, and inversely (B, T_B) can be seen as a Poincare section of the geodesic flow.
We proved that this geodesic billiard is actually conjugated with the extension T_C of T_L and T_R. Moreover, that conjugacy can be defined on a finite partition in blocks of B and acts diagonally by some isometry of Γ on any of these blocks :

Theorem 2.2. *(B, T_B) and (C, T_C) are conjugated by a bijection $\varphi : B \to C$ such that :*
(i) $\varphi = id$ on $B \cap C$;
(ii) there are $p > 0$, $X_1 \dots X_p \subset B$ and $\gamma_1 \dots \gamma_p \in \Gamma$ for which :
– $B \setminus C = \sqcup_{k=1}^p X_i$;
– $C \setminus B = \sqcup_{k=1}^p Y_i$ where $Y_i = \gamma_i(X_i)$;
– $\forall i, \forall (x, y) \in X_i, \varphi(x, y) = (\gamma_i(x), \gamma_i(y))$.

3. Transfer operator and eigenfunctions of the hyperbolic laplacian

Consider :
(i) $P^s(z, x) = \left(\frac{1-|z|}{|z-x|}\right)^s$ the usual Poisson kernel ;
(ii) $\mathcal{E}_\lambda^e$ the space of the eigenfunctions of the hyperbolic laplacian on $\mathbb{D}$ for the eigenvalue λ that are at most of exponential growth in the hyperbolic radius ;
(iii) $\mathcal{D}'(\mathbb{S}^1)$ the space of distributions over $\mathbb{S}^1$.
A well-known result of Helgason states that you can represent every function f of $\mathcal{E}_\lambda^e$ by a couple of distributions $\mathcal{D}_{f,s}$ and $\mathcal{D}_{f,1-s}$ on $\mathbb{S}^1$, the *Helgason boundary values* of f :

Theorem 3.1 (Helgason).

$$\mathcal{P}^s : \begin{array}{rcl} \mathcal{D}'(\mathbb{S}^1) & \to & \mathcal{E}_{s(1-s)}^e \\ \nu & \mapsto & (z \mapsto \langle \nu, P^s(z, .)\rangle) \end{array}$$

is a continuous isomorphism with inverse $f \to \mathcal{D}_{f,s}$.

When the eigenfunctions are bounded, Otal refined this result in [2]. First, we define the *derivative* of a continuous function F over $[0; 2\pi[$ by the linear functional :

$$F' : \begin{array}{rcl} \mathcal{C}^1(\mathbb{S}^1) & \to & \mathbb{C} \\ \varphi & \mapsto & (F(2\pi) - F(0))\varphi(0) - \int_0^{2\pi} \varphi'(t)F(t)dt \end{array}.$$

Then take :
(i) $\mathcal{E}_\lambda^b$ the subspace of bounded functions of $\mathcal{E}_\lambda^e$;
(ii) Λ_α the space of α-Hölder functions over $[0; 2\pi[$ that vanish at 0 ;
(iii) Λ_α^1 the space of derivates of such functions.

Theorem 3.2 (Otal). *Assume $0 < \Re(s) \leq 1$. Then :*

$$\mathcal{P}^s : \begin{array}{rcl} \Lambda_{\Re(s)}^1 & \to & \mathcal{E}_{s(1-s)}^b \\ \nu & \mapsto & (z \mapsto \langle \nu, P^s(z, .)\rangle) \end{array}$$

is a continuous isomorphism with inverse $f \to \mathcal{D}_{f,s}$. More precisely, if $f = \mathcal{P}^s(D')$ with $D \in \Lambda_\alpha$, then

$$\forall z \in \mathbb{D}, |f(z)| \leq C(s)\|D\|_\alpha e^{-(\alpha - \Re(s))d(0,z)}.$$

This implies that if f is a bounded solution of $\Delta f = s(1-s)f$ that is also automorphic for a cofinite group Γ, then $\Delta_{f,s}$ is the derivative of a $\Re(s)$-Hölder function **and nothing more**.
The *transfer operator* of $T = T_L$ or T_R is given by :

$$\mathcal{L}_s : \begin{array}{rcl} E & \to & E \\ f & \mapsto & \left(y \in \mathbb{S}^1 \mapsto \displaystyle\sum_{T(x)=y} \frac{f(x)}{|T'(x)|^s}\right) \end{array}$$

It acts on the space E of complex functions defined on $\mathbb{S}^1$ that are $\mathcal{C}^1$ on every $\overline{I_k^L} = \overline{I_k^R}$.
We proved in [3] that the eigendistributions of this operator for the eigenvalue 1 are exactly the Helgason boundary values of automorphic eigenfunctions of the hyperbolic laplacian :

Theorem 3.3. *Let $\nu \in \Lambda_{\Re(s)}^1$. Then $\mathcal{L}_{s,R}^\star \nu = \nu$ if and only if $\mathcal{P}^s(\nu) \in \mathcal{E}_{s(1-s)}^b$ is Γ-automorphic.*

References

[1] R. Bowen and C. Series. Markov Maps associated with Fuschians Groups. *Publications de l'IHES*, 50:401–418, 1979.

[2] J.-P. Otal. Sur les fonctions propres du Laplacien du disque hyperbolique. *Comptes-Rendus de l'Académie des Sciences*, 327(1):161–166, 1998.

[3] V. Pit. Invariant relations for the Bowen-Series transform. *Conformal Geometry and Dynamics*, 16:103–123, 2012.

[4] M. Pollicott. Some Applications of Thermodynamic Formalism to Manifolds with Constant Negative Curvature. *Advances in Mathematics*, 85:161–192, 1991.