

# A geometrical approach to Bowen-Series coding of the geodesic flow on hyperbolic surfaces of finite volume

Vincent Pit

Institut de Mathématiques de Bordeaux  
Université Bordeaux 1, 351 cours de la Libération, 33405 Talence Cedex, France

## 1. Bowen-Series coding

This coding is a combinatorial object that was introduced in [2]. Its dynamical properties hold strong relations with the geodesic flow on hyperbolic surfaces.

### 1.1 Construction

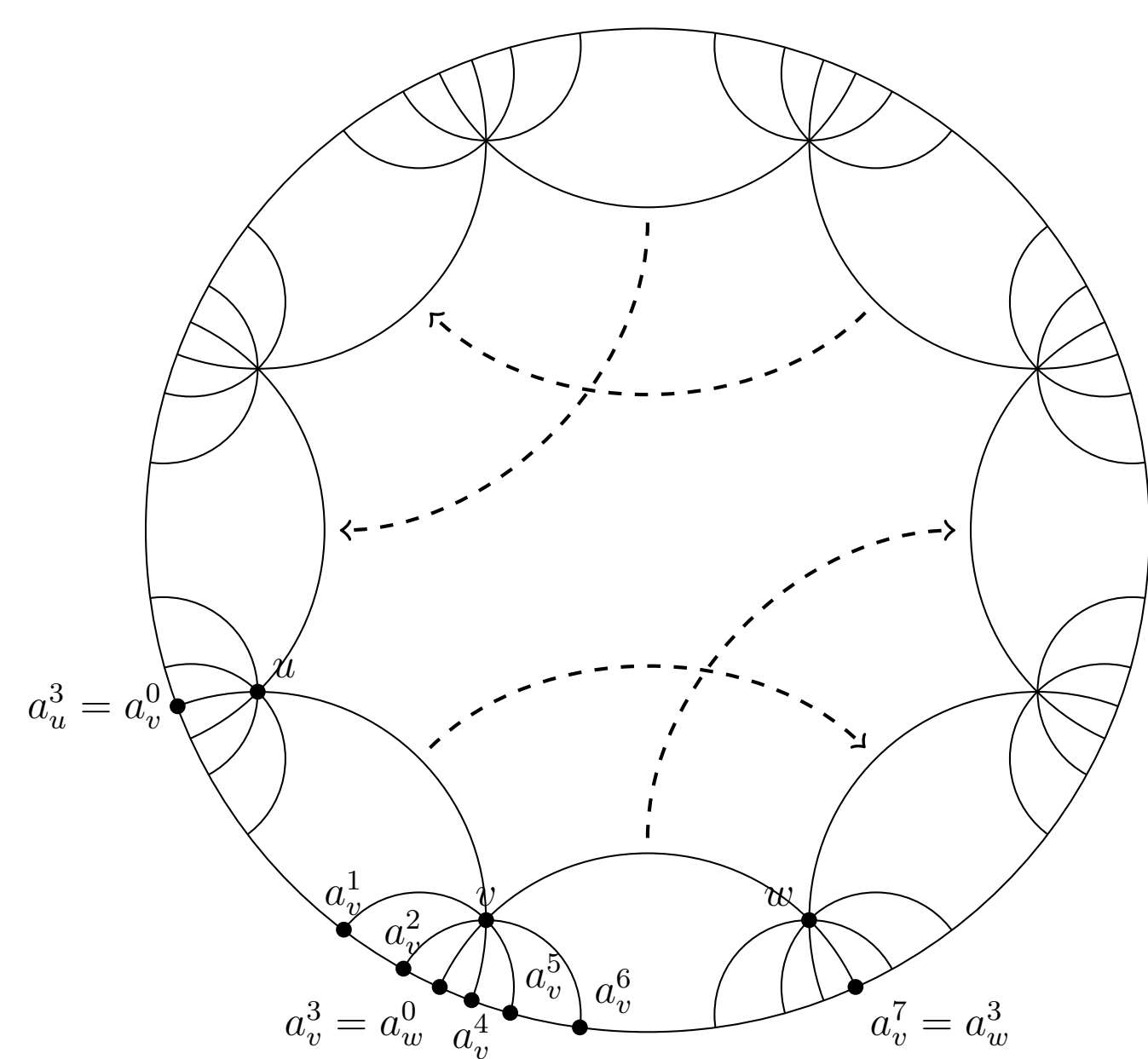
We denote  $\mathbb{D}$  the Poincaré disk and  $\partial\mathbb{D} \simeq \mathbb{S}^1$  its boundary. Let  $\Gamma$  be a cofinite fuchsian group that isn't a triangle group, and that admits a fundamental domain  $\mathcal{D}$  satisfying these properties :

- $\mathcal{D}$  is geodesically convex ;
- its boundary is a finitely-sided geodesic polygon with sides  $E$  and vertices  $V \subset \mathbb{D} \cup \partial\mathbb{D}$  ;
- for every  $e \in E$ , there exists  $\gamma_e \in \Gamma \setminus \{id\}$  such that  $\gamma_e(e) \in E$  ; and  $f = \gamma_e(e)$  if and only if  $\gamma_f(f) = e$  ;
- $(\gamma_e)_{e \in E}$  generates  $\Gamma$ .

For example, we can take a Dirichlet domain based at any  $P \in \mathbb{D}$  that isn't the center of an elliptic isometry of  $\Gamma$ . Each side  $e \in E$  can be extended to a unique geodesic  $\tilde{e}$ . Let  $\mathcal{N}$  be the network of geodesics  $\gamma(\tilde{e})$  where  $\gamma \in \Gamma$ ,  $e \in E$  and  $\gamma(\tilde{e})$  either passes by an inner vertex or by two distinct vertices of  $\mathcal{D}$ . We make the following even corners assumption : **no geodesic in  $\mathcal{N}$  crosses the interior of the fundamental domain.**

The trace of  $\mathcal{N}$  on  $\partial\mathbb{D}$  is a finite set that delimits a finite partition in intervals. For  $v \in V$ , we name the endpoints of geodesics of  $\mathcal{N}_v = \{g \in \mathcal{N} \mid v \in g\}$  :

- when  $v$  is an inner vertex,  $\mathcal{N}_v$  contains  $m_v = 2n_v$  elements. We start by calling  $a_v^0$  the furthest endpoint of the left edge that goes through  $v$ , and likewise  $a_v^{m_v-1}$  for the right edge. By the even corners hypothesis, all other endpoints are in  $]a_v^0; a_v^{m_v-1}[$  ; so we order them  $a_v^k$  for  $0 < k < m_v - 1$ .
- when  $v$  is on the boundary, we artificially set  $n_v = 3$ , so that  $a_v^0 = a_v^2 = a_v^3 = a_v^4 = v$ .



For  $v \in V$ , let  $\gamma_v$  be the generator associated with the edge of the domain at the right of  $v$ ,  $I_{v,L} = [a_v^{n_v}; a_v^{n_w}]$  and  $I_{v,R} = ]a_v^{n_w-1}; a_v^{n_v-1}[$ . We define the *left and right*

*Bowen-Series transformations* by :

$$\begin{aligned} T_L : \mathbb{S}^1 &\rightarrow \mathbb{S}^1 \\ x \in I_{v,L} &\mapsto \gamma_v(x) \\ T_R : \mathbb{S}^1 &\rightarrow \mathbb{S}^1 \\ x \in I_{v,R} &\mapsto \gamma_v(x) \end{aligned}$$

$T_L$  and  $T_R$  are Markov on the partition of  $\mathbb{S}^1$  delimited by all the  $a_v^k$ .

For all  $k \geq 0$ , we can also recursively define the *left and right words* in the Bowen-Series coding by :

$$\begin{aligned} \gamma_L^k : \mathbb{S}^1 &\rightarrow \Gamma \\ x \in I_{v,L} &\mapsto \gamma_L^k[x] = \gamma_L^{k-1}[T_L(x)]\gamma_v \\ \gamma_R^k : \mathbb{S}^1 &\rightarrow \Gamma \\ x \in I_{v,R} &\mapsto \gamma_R^k[x] = \gamma_R^{k-1}[T_R(x)]\gamma_v \end{aligned}$$

### 1.2 Fundamental lemma

In all the following,  $T$  denotes either  $T_L$  or  $T_R$ .

The lemma we're going to state encompasses all the dynamical properties of  $T$ . Most of the subsequent results are corollaries of this one.

Note  $\mathbb{I} = \{e^{ia}; e^{ib} \mid 0 \leq a \leq b \leq 2\pi\}$ . Let  $X$  be a set on which  $\Gamma$  acts on the left.  $\Gamma$  acts also naturally on the left on  $\mathbb{I}$ .

We say that  $F : \mathbb{I} \times X \rightarrow \mathbb{C}$  verifies  $\mathcal{I}(I, \gamma)$  when :

$$\forall x \in X, F(I, x) = F(\gamma(I), \gamma(x)).$$

**Lemma 1.2.1.** *Let  $F : \mathbb{I} \times X \rightarrow \mathbb{C}$  such that*

- if  $F$  verifies  $\mathcal{I}(I, \gamma)$ , then it verifies  $\mathcal{I}(J, \gamma)$  for all  $J \subset I$ ,  $J \in \mathbb{I}$  (inclusion) ;*
- if  $I, J, I \sqcup J \in \mathbb{I}$ , then  $F(I \sqcup J, x) = F(I, x) + F(J, x)$  for all  $x \in X$  (additivity for contiguous intervals) ;*
- if  $(b_n) \rightarrow b$ , then  $(F(\cdot; b_n), x) \rightarrow F(\cdot; b), x$  for all  $x \in X$  (continuity).*

*(iv)  $F$  verifies  $\mathcal{I}(I_v, \gamma_v)$  for every  $v \in V$ .*

*Then  $F$  verifies  $\mathcal{I}(I, \gamma)$  for every  $I \in \mathbb{I}$  and  $\gamma \in \Gamma$ .*

Basically, it allows us to transport the combinatorics of intervals under the action of  $T$  to relations in  $X$ .

### 1.3 Periodic points and hyperbolic isometries of $\Gamma$

The lemma can easily prove this famous result :

**Theorem 1.3.1** (Series, [2]).  *$T$  is orbit-equivalent with the group  $\Gamma$ , i.e. for all  $x, y \in \mathbb{S}^1$ ,*

$$\exists \gamma \in \Gamma, y = \gamma(x) \Leftrightarrow \exists p, q \geq 0, T^p(x) = T^q(y).$$

Since  $\Gamma$  is a group of the first kind, this implies :

**Theorem 1.3.2** (Pre-periodic points).

- If  $y \in \text{Per}(T)$ , then  $\{x \in \mathbb{S}^1 \mid \exists p \geq 0, T^p(x) = y\}$  is dense in  $\mathbb{S}^1$ .*
- $\{x \in \mathbb{S}^1 \mid \exists p \geq 0, T^p(x) \in \text{Per}(T)\} \subset \text{Fix}(\Gamma)$ .*

The fact that pre-images of a periodic points are dense forbids us to be able to find a trivial word in the coding :

**Theorem 1.3.3.** *For all  $x \in \mathbb{S}^1$  and  $k > 0$ ,  $\gamma^k[x] \neq id$ .*

The fundamental lemma can actually prove a stronger variant of the orbit-equivalence theorem :

**Theorem 1.3.4** (Series, revisited).  *$(x, k) \rightarrow \gamma^k[x]$  is orbit-equivalent with the group  $\Gamma$ , i.e.*

$$\forall x \in \mathbb{S}^1, \forall \gamma \in \Gamma, \exists p, q \geq 0, \gamma^p[x] = \gamma^q[\gamma(x)]\gamma.$$

This allows us to prove the other inclusion :

**Theorem 1.3.5** (Pre-periodic points, revisited).

*(i)  $\{x \in \mathbb{S}^1 \mid \exists p \geq 0, T^p(x) \in \text{Per}(T)\} = \text{Fix}(\Gamma)$ .*

*(ii) If  $x \in \text{Per}(T)$  has period  $k$ , then  $\gamma^k[x]$  is primitive.*

With this, we're now in position to identify periodic  $T$ -orbits with conjugacy classes of primitive elements of  $\Gamma$  under the extra hypothesis that  $|T'| \geq 1$ . This is verified in particular when edges of the fundamental domain are the isometric circles of their associated generators, e.g. for a Dirichlet domain centered at 0.

**Theorem 1.3.6.** *Suppose that  $|T'| \geq 1$ . There is a bijection between periodic hyperbolic orbits of  $T$  and conjugacy classes of primitive hyperbolic elements of  $\Gamma$ .*

**Corollary 1.3.7.** *Suppose that  $|T'| \geq 1$ . There is a bijection between periodic orbits of  $T$  that don't pass through a cusp of  $\Gamma$  and conjugacy classes of primitive hyperbolic elements of  $\Gamma$ .*

Morita gives a similar result in [4] but a finite number of periodic orbits are missing from the counting.

## 2. Natural extension

One can wonder what is the relation between the Bowen-Series coding and the geodesic flow on  $\mathbb{D}/\Gamma$ . We link them through the *geodesic billiard* of  $\mathcal{D}$ .

### 2.1 Construction

$T_L$  and  $T_R$  can respectively be seen as the first factor of a Baker-like transform and the right factor of its inverse.

**Theorem 2.1.1.** *Let  $\Delta$  be the diagonal the 2-torus  $\mathbb{T}^2$ . There exists  $C \subset \mathbb{T}^2 \setminus \Delta$  and  $T_C : C \rightarrow C$  such that :*

- $T_C$  is a bijection ;*
- For every  $(x, y) \in C$ ,*

$$T_C(x, y) = (\gamma_R[y](x), \gamma_R[y](y)) = (S_L(x, y), T_R(y))$$

$$T_C^{-1}(x, y) = (\gamma_L[x](x), \gamma_L[x](y)) = (T_L(x), S_R(x, y));$$

$$(iii) T_C^p(x, y) = (x, y) \Leftrightarrow T_L^p(x) = x \Leftrightarrow T_R^p(y) = y.$$

A first construction of an extension of the coding was given in [1], but only in the cocompact case and for a specific fundamental domain.

### 2.2 Extension and geodesic billiard

Let  $B$  be the set of all geodesics of  $\mathbb{D}$  that either

- cross the interior of  $\mathcal{D}$  ;
- pass through a vertex  $v \in V$  while keeping the fundamental domain on their right ;
- are in  $\mathcal{N}$  and keep the domain on their left.

We set  $T_B(x, y) = (\gamma_e(x), \gamma_e(y))$  whenever  $(x, y) \in B$  leaves the fundamental domain by the edge  $e$ .  $T_B$  is a bijection of  $B$  and  $(B, T_B)$  is called the *geodesic billiard*. The geodesic flow on  $\mathbb{D}/\Gamma$  can be obtained as the suspension of  $(B, T_B)$  by the transit time, and inversely  $(B, T_B)$  can be seen as a Poincaré section of the geodesic flow.

**Theorem 2.2.1.**  *$(B, T_B)$  and  $(C, T_C)$  are conjugated. More precisely, we can construct a bijection  $\varphi : B \rightarrow C$  such that :*

- $\varphi = id$  on  $B \cap C$  ;*
- there exists  $p > 0$ ,  $X_1 \dots X_p \subset \mathbb{T}^2 \setminus \Delta$  and  $\gamma_1 \dots \gamma_p \in \Gamma$  for which :*  
–  $B \setminus C = \sqcup_{k=1}^p X_k$  ;  
–  $C \setminus B = \sqcup_{k=1}^p Y_k$  where  $Y_k = \gamma_k(X_k)$  ;  
–  $\forall i, \forall (x, y) \in X_i, \varphi(x, y) = (\gamma_i(x), \gamma_i(y))$ .
- $\varphi T_B = T_C \varphi$ .*

The most interesting fact is that the conjugacy is determined by the given of a finite partition.

### 2.3 Bordering geodesics

From the previous study of the coding, we can describe what are the geodesics that edge the fundamental domain.

**Theorem 2.3.1.** *Let  $g$  be a geodesic of  $\mathbb{D}$  that borders the fundamental domain  $\mathcal{D}$ . Then :*

- either  $g$  is the axis of an hyperbolic element of  $\Gamma$  and thus projects itself onto a closed geodesic of  $\mathbb{D}/\Gamma$  ;*
- or both endpoints of  $g$  are projected onto (possibly different) cusps of  $\mathbb{D}/\Gamma$ .*

## 3. Transfer operator and eigenfunctions of the hyperbolic laplacian

### 3.1 Helgason boundary values

Consider :

- $\mathcal{E}_\lambda$  the space of the eigenfunctions of the hyperbolic laplacian on  $\mathbb{D}$  for the eigenvalue  $\lambda$  ;
- $\mathcal{E}_\lambda^c$  the subspace of those that are at most of exponential growth in the hyperbolic radius ;
- $\mathcal{D}'(\mathbb{S}^1)$  the space of distributions over  $\mathbb{S}^1$ .

A well-known result says that you can represent every function  $f$  of  $\mathcal{E}_\lambda^c$  by a couple of distributions  $\mathcal{D}_{f,s}$  and  $\mathcal{D}_{f,1-s}$  on  $\mathbb{S}^1$ , the *Helgason boundary values* of  $f$ .

**Theorem 3.1.1** (Helgason).

$$\begin{aligned} \mathcal{P}^s : \mathcal{D}'(\mathbb{S}^1) &\rightarrow \mathcal{E}_{-s(1-s)}^c \\ T &\mapsto z \mapsto \langle T, \mathcal{P}^s(z, \cdot) \rangle \end{aligned}$$

*is a continuous isomorphism of reciprocal  $f \mapsto \mathcal{D}_{f,s}$ .*

When the eigenfunctions are bounded, Otal refined this result in [5]. First, we define the *derivative* of a continuous function  $F$  defined over  $[0; 2\pi[$  by the linear functional

$$F' : \mathcal{C}^1(\mathbb{S}^1) \rightarrow \mathbb{C} \\ \varphi \mapsto (F(2\pi) - F(0))\varphi(0) - \int_0^{2\pi} \varphi'(t)F(t)dt$$

Then take :

- $\mathcal{E}_\lambda^b$  the subspace of bounded functions of  $\mathcal{E}_\lambda$  ;
- $\Lambda_\alpha$  the space of  $\alpha$ -Hölder functions over  $[0; 2\pi[$  that vanish at 0 ;
- $\Lambda_\alpha^0$  the space of derivatives of such functions.

**Theorem 3.1.2** (Otal).

$$\begin{aligned} \mathcal{P}^s : \Lambda_{\mathbb{R}(s)}^1 &\rightarrow \mathcal{E}_{-s(1-s)}^b \\ T &\mapsto z \mapsto \langle T, \mathcal{P}^s(z, \cdot) \rangle \end{aligned}$$

*is a continuous isomorphism of reciprocal  $f \mapsto \mathcal{D}_{f,s}$ . More precisely, if  $f = \mathcal{P}^s(D')$  with  $D \in \Lambda_\alpha$ , then*

$$\forall z \in \mathbb{D}, |f(z)| \leq C(s)\|D\|_\alpha e^{-(\alpha - \Re(s))d(0,z)}.$$

This implies that if  $f$  is a bounded solution of  $\Delta f = -s(1-s)f$  that is also automorphic for a cofinite group  $\Gamma$ , then  $\Delta_{f,s}$  is the derivative of a  $\mathfrak{R}(s)$ -Hölder function **and nothing more.**

### 3.2 Eigenfunctions and eigendistributions of the transfer operator

The *transfer operator* of  $T = T_L$  or  $T_R$  is given by :

$$\begin{aligned} \mathcal{L}_s : E &\rightarrow E \\ f &\mapsto \mathcal{L}_s(f) : y \in \mathbb{S}^1 \mapsto \sum_{T(x)=y} \frac{f(x)}{|T'(x)|^s} \end{aligned}$$

The eigendistributions of this operator for the eigenvalue 1 are exactly the Helgason boundary values of eigenfunctions of the hyperbolic laplacian :

**Theorem 3.2.1.** *Let  $\nu \in \Lambda_{\mathbb{R}(s)}^1$ . Then  $\mathcal{L}_{s,R}^* \nu = \nu$  if and only if  $\mathcal{P}^s(\nu) \in \mathcal{E}_{-s(1-s)}^b$  is  $\Gamma$ -automorphic.*

This result was hinted in [6] for a different setting.

It's natural to focus now on the 1-eigenfunctions of the transfer operator. Note  $d(x, y) = \frac{|x-y|}{2}$  the Gromov distance on  $\mathbb{S}^1$  and  $k^s(x, y) = d(x, y)^{-2s}$ . The only thing we know about the eigenfunctions is that there are more of them than eigendistributions :

**Theorem 3.2.2** (Lopes-Thieullen, [3]). *Suppose  $\Gamma$  cocompact. Take  $\mathcal{D}_{f,s}$  the boundary value of  $f \in \mathcal{E}_{-s(1-s)}^b$   $\Gamma$ -invariant. Let  $\psi_{f,s}(x) = \langle \mathcal{D}_{f,s}, k^s(x, \cdot) \mathbb{1}_C(x, \cdot) \rangle$  where  $C$  is the support of the natural extension  $(C, T_C)$  of  $T_L$  and  $T_R$ . Then  $\mathcal{L}_{s,L} \psi_{f,s} = \psi_{f,s}$  and  $\mathcal{D}_{f,s} \mapsto \psi_{f,s}$  is injective.*

## References

- [1] R. Adler and L. Flatto. Geodesic Flows, Interval Maps and Symbolic Dynamics. *Bulletin of the American Mathematical Society*, 25(2):229–334, 10 1991.
- [2] R. Bowen and C. Series. Markov Maps associated with Fuchsians Groups. *Publications de l'IHES*, 50:401–418, 1979.
- [3] A. O. Lopes and P. Thieullen. Eigenfunctions of the Laplacian and eigenfunctions of the associated Ruelle operator. *Nonlinearity*, 21:2239–2253, 2008.
- [4] T. Morita. Markov systems and transfer operators associated with cofinite Fuchsian groups. *Ergodic Theory and Dynamical Systems*, 17:1147–1181, 1997.
- [5] J.-P. Otal. Sur les fonctions propres du Laplacien du disque hyperbolique. *Comptes-Rendus de l'Académie des Sciences*, 327(1):161–166, 1998.
- [6] M. Pollicott. Some Applications of Thermodynamic Formalism to Manifolds with Constant Negative Curvature. *Advances in Mathematics*, 85:161–192, 1991.