Exercises:

In the following exercises, we study the geodesics of metrics defined over open subsets of $\mathbb{R}^2$.

1: The spherical metric in the stereographic projection is defined over the whole of $\mathbb{R}^2$ and is given by

$$g_{ij} = \frac{4}{(1+r^2)^2}\delta_{ij}.$$

a: Show that, up to reparametrisation, for all θ, the curve

$$c(t) = te^{i\theta}$$

is a geodesic of this metric.

b: Show that, up to reparametrisation, for all θ, and for all $r > 0$, the curve

$$c(t) = \frac{r^2-1}{2r}e^{i\theta} + \frac{r^2+1}{2r}e^{i(\theta+t)}$$

is a geodesic of this metric.

c: Show that the curve described in (b) is a circle whose nearest point to the origin lies at a distance of $\frac{1}{r}$ and whose furthest point from the origin lies at a distance of r.

d: Show that this metric has no other geodesics.

2: The hyperbolic metric in the Poincaré disk model is defined over the disk $\{|z| < 1\}$ and is given by

$$g_{ij} = \frac{4}{(1-r^2)^2}\delta_{ij}.$$

a: Show that, up to reparametrisation, for all θ, the curve

$$c(t) = te^{i\theta}$$

is a geodesic of this metric.

b: Show that, up to reparametrisation, for all θ, and for all $r > 1$, the curve

$$c(t) = re^{i\theta} + \sqrt{r^2-1}e^{i(\theta+t)}$$

is a geodesic of this metric.

c: Show that the curve described in (b) is a circle which intersects the unit circle orthogonally.

d: Show that this metric has no other geodesics.

3: The hyperbolic metric in the upper half-space model is defined over the half-space $\{x + iy \mid y > 0\}$ and is given by

$$g_{ij} = \frac{1}{y^2}\delta_{ij}.$$

a: Show that, up to reparametrisation, for all x, the curve

$$c(t) = x + it$$

is a geodesic of this metric.

b: Show that, up to reparametrisation, for all x, and for all $r > 0$, the curve

$$c(t) = x + re^{i\theta}$$

is a geodesic of this metric.

c: Show that the curve described in (b) is a circle which intersects the real axis orthogonally.

d: Show that this metric has no other geodesics.