(1) Let $\Omega \subseteq \mathbb{R}^m$ be open. Denote $\chi(\Omega) := C^\infty(\Omega, \mathbb{R}^m)$. Let $\nabla : \chi(\Omega) \oplus \chi(\Omega) \to \chi(\Omega)$ be a covariant derivative. Show that there exist smooth functions $\Gamma^k_{ij} : \Omega \to \mathbb{R}$ such that, for all $\xi, \eta \in \chi(\Omega)$,

$$(\nabla_\xi \eta)^i = \xi^j \frac{\partial \eta^i}{\partial x_j} + \Gamma^i_{jk} \xi^j \eta_k.$$

In particular, $(\nabla_\xi \eta)(x)$ only depends on $\xi(x)$, $\eta(x)$ and $(D_\xi \eta)(x)$. Show that ∇ is torsion free if and only if, for all i, j, k,

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

(2) Let $\Omega \subseteq \mathbb{R}^m$ be open. Let g be a metric over Ω. Let $\gamma :]a, b[\to \Omega$ be a geodesic. Show that

$$g(\dot\gamma, \dot\gamma) = g(\gamma(t))_{ij} \dot\gamma(t)^i \dot\gamma(t)^j$$

is constant. Let $\xi, \eta :]a, b[\to \mathbb{R}^m$ be parallel vector fields over Γ. Show that

$$g(\xi, \eta) = g(\gamma(t))_{ij} \xi(t)^i \eta(t)^j$$

is constant.

(3) Let $\Omega \subseteq \mathbb{R}^m$ be open. Recall that Ω is said to be convex whenever it has the property that, for all $x, y \in \Omega$ and for all $t \in [0, 1]$,

$$tx + (1 - t)y \in \Omega.$$

Define $\delta : \Omega \times \Omega \to \mathbb{R}^m$ by

$$\delta(x, y) := \operatorname*{Inf}_\gamma \int_0^1 \|\dot\gamma(t)\| dt,$$

where the infimum is taken over all C^1 curves $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Show that δ defines a metric (distance function) over Ω. Show that

$$\delta(x, y) \geq \|x - y\|.$$

Show that $\delta(x, y) = \|x - y\|$ for all x, y if and only if Ω is convex.

(4) Let $\Omega \subseteq \mathbb{R}^m$ be open. For $f \in C^2(\Omega, \mathbb{R})$, define

$$\mathrm{Hess}(f) := \nabla df.$$

Show that, for all vector fields ξ and η,

$$\mathrm{Hess}(f)(\xi, \eta) = D_\xi(df(\eta)) - df(\nabla_\xi \eta).$$

Show that, for all vector fields ξ and η,

$$\mathrm{Hess}(f)(\xi, \eta) = \frac{\partial^2 f}{\partial x_i \partial x_j} \xi^i \eta^j - \Gamma^k_{ij} \xi^i \eta^j \frac{\partial f}{\partial x_k}.$$

Show that $\mathrm{Hess}(f)$ is symmetric.

(5) Let $X \subseteq \mathbb{R}^m$ be a submanifold. Let $\phi : X \to X$ be a smooth isometry. Let

$$Y := \{y \in X \mid \phi(y) = y\}.$$

Suppose that Y is a submanifold of X. Show that Y is totally geodesic.

Show that $\mathrm{O}(m)$ is a submanifold of $\mathrm{End}(\mathbb{R}^m)$. Choose p, q such that $p + q = m$. Denote

$$M := \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & -\mathrm{I}_q \end{pmatrix}.$$

Define $\phi : \mathrm{O}(m) \to \mathrm{O}(m)$ by

$$\phi(N) := M^{-1}NM.$$

Show that ϕ is an isometry. Show that

$$Y = \left\{ \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \;\middle|\; N_1 \in \mathrm{O}(p),\ N_2 \in \mathrm{O}(q) \right\}.$$

It follows that Y is totally geodesic.

(6) Let Ω be an open subset of $\mathbb{R}^m$. Let g be a metric over Ω. Let $X \subseteq \Omega$ be a submanifold. Show that X is totally geodesic with respect to g if and only if its second fundamental form vanishes.

(7) Let g be a metric over $\mathbb{R}^3$ with the following properties

(a) the z-axis is a unit speed geodesic;

(b) all horizontal radial lines are unit speed geodesics;

(c) all horizontal radial lines are orthogonal to the z-axis; and

(d) g has constant sectional curvature equal to -1.

Show that

$$g = dr^2 + \sinh^2(r)d\theta^2 + \cosh^2(r)dz^2.$$

(8) Let g be a metric over $\mathbb{R}^3$ with the following properties

(a) the $x - y$ plane is totally geodesic;

(b) all radial lines in the $x - y$ plane are unit speed geodesics;

(c) all vertical lines are unit speed geodesics;

(d) all vertical lines are orthogonal to the $x - y$ plane; and

(e) g has constant sectional curvature equal to -1.

Show that

$$g = \cosh^2(z)dr^2 + \cosh^2(z)\sinh^2(r)d\theta^2 + dz^2.$$

(9) Repeat exercises (7) and (8) supposing now that the sectional curvature is equal to +1. Show that in this case the domains of g are

$$\left\{(x,y,z)\mid\sqrt{x^2+y^2}<\pi\right\}$$

and

$$\left\{(x,y,z)\mid\sqrt{x^2+y^2}<\pi, |z|<\pi\right\}$$

respectively.

(10) Let $\Omega\subseteq\mathbb{R}^m$ be open. Let g be a metric over Ω. Let $f:\Omega\to\mathbb{R}$ be a submersion. Let

$$X:=f^{-1}(\{0\}).$$

Show that the unit normal vector field over X is

$$N:=\frac{1}{\|\nabla f\|}\nabla f.$$

Show that the second fundamental form of X is related to the Hessian of f by

$$\mathrm{II}_X(x)=\frac{1}{\|\nabla f(x)\|}\mathrm{Hess}(f)(x)|_{T_xX}.$$

Consider another smooth function $g:\Omega\to\mathbb{R}$. Show that the Hessian of the restriction of this function to X satisfies

$$\mathrm{Hess}_X\left(g|_X\right)=\mathrm{Hess}(g)|_{TX}-\frac{1}{\|\nabla f\|}\langle\nabla f,\nabla g\rangle\mathrm{II}_X.$$

(11) Let $\Omega\subseteq\mathbb{R}^m$ be open. Let g be a metric over Ω. Let $\gamma:]a,b[\to\Omega$ be a geodesic. Suppose that $0\in]a,b[$. Show that points conjugate to 0 along γ are isolated.

(12) Consider the metric over $\mathrm{End}(\mathbb{R}^2)$ given by

$$\langle A,B\rangle:=\mathrm{Tr}(AB^t).$$

Define

$$\Sigma:=\left\{A\in\mathrm{End}(\mathbb{R}^2)\mid\|A\|=1\right\}.$$

Define

$$T:=\{A\in\Sigma\mid\mathrm{Rk}(A)=1\}.$$

Show that T is a minimal surface in Σ isometric to the torus

$$\frac{1}{\sqrt{2}}(S^1\times S^1).$$

Define

$$C := \left\{A \in \Sigma \mid A^t A = 1/2\mathrm{Id}\right\}.$$

Show that C is a 1-dimensional submanifold of Σ isometric to S^1. Show that

$$\operatorname*{Inf}_{A\in C, B\in T} d(A,B) = \frac{\pi}{4}.$$

Show, furthermore, that for all $A \in C$, there exists $B \in T$ such that

$$d(A,B) = \frac{\pi}{4}.$$

Likewise, for all $A \in T$, there exists $B \in C$ such that

$$d(A,B) = \frac{\pi}{4}.$$

T is a model of the Clifford torus in S^3.

(13) Let Ω be an open convex subset with smooth boundary. Show that, for all $x \in \mathbb{R}^m$ there exists a unique point $\pi(x) \in \overline{\Omega}$ such that

$$\|x - \pi(x)\| = \operatorname*{Inf}_{z\in\Omega} \|x - z\|.$$

Show, furthermore, that if $x \notin \Omega$, then $\pi(x) \in \partial\Omega$. Show that, for all $x, y \in \mathbb{R}^m$,

$$\|\pi(x) - \pi(y)\| \leq \|x - y\|.$$

Show that π is continuous. Let $D : \mathbb{R}^m \to \mathbb{R}$ denote the distance to Ω, that is

$$D(x) := \|x - \pi(x)\|.$$

Show that D is smooth over $\mathbb{R}^m \setminus \Omega$. Show that, for all $x \in \mathbb{R}^m \setminus \overline{\Omega}$,

$$\nabla D(x) = \frac{x - \pi(x)}{\|x - \pi(x)\|}.$$

Show that D is convex over $\mathbb{R}^m$.

Observe that $\partial\Omega$ is a submanifold of $\mathbb{R}^m$. Let $\delta : \partial\Omega \times \partial\Omega \to \mathbb{R}$ be its riemannian distance. Show that, for all $x, y \in \mathbb{R}^m \setminus \Omega$ such that

$$\{tx + (1-t)y \mid t \in [0,1]\} \cap \Omega = \emptyset,$$

we have

$$\delta(\pi(x), \pi(y)) \leq \|x - y\|.$$

(14) Consider the metric g over $\mathbb{R}^m$ given by

$$g_{ij} = \frac{1}{x_m}\delta_{ij}.$$

Show that g is not complete.

(15) Consider the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ defined over $\mathbb{R}^{m+1}$ by

$$\langle \xi, \eta \rangle := x^1 y^1 + \dots + x^m y^m - x^{m+1} y^{m+1}.$$

Define

$$X := \left\{\xi \in \mathbb{R}^{m+1} \mid \langle \xi, \xi \rangle = -1,\ x_{m+1} > 0\right\}.$$

Show that X is a submanifold of $\mathbb{R}^{m+1}$. Show that $\langle \cdot, \cdot \rangle$ defines a riemannian metric over X. Show that this metric is complete and has constant sectional curvature equal to -1. X is a model of m-dimensional hyperbolic space.