Riemannian Geometry - Exam 10/12/2015

Question A: Consider the metric, g_{ij}, defined over $\mathbb{R}^2$ by

$$g_{ij} = e^{-(x^2+y^2)}\delta_{ij}.$$

Calculate the scalar curvature of g.

Question B: Consider the constant metric, g, defined over $\mathbb{R}^4$ by

$$g(u,v) = u_1v_1 + u_2v_2 + u_3v_3 - u_4v_4.$$

Anti de-Sitter space is defined by $X := \left\{x \in \mathbb{R}^4 \mid g(x,x) = 1\right\}$. Show that X is a manifold. Show that the restriction of g to X has signature $(2,1)$. Calculate the shape operator of X. Calculate the sectional curvature of X.

N.B. Special care should be taken with the sign, since the metric is not positive definite.

Question C: Consider the constant metric, g, defined over $\mathbb{R}^3$ by

$$g(u,v) = u_1v_1 + u_2v_2 - u_3v_3.$$

Hyperbolic space is defined by $X := \left\{x \in \mathbb{R}^3 \mid g(x,x) = -1,\ x_3 > 0\right\}$. Show that X is a manifold. Let D be the unit disk in $\mathbb{R}^2$, and consider the map $\Phi : X \to D$ given by

$$\Phi(x) = \frac{(x_1,x_2)}{x_3}.$$

Show that Φ is a diffeomorphism. Show that Φ sends geodesics in X to straight lines in D.