**Riemannian Geometry - Exam 10/12/2015**

**Question A:** Consider the Grim metric, $g_{ij}$, defined over $\mathbb{R}^3$ by

$$g_{ij} = e^{-y}\delta_{ij}.$$

Calculate the Ricci curvature tensor of $g$.

**Question B:** Consider the constant metric, $g$, defined over $\mathbb{R}^4$ by

$$g(u,v) = u_1v_1 + u_2v_2 - u_3v_3 - u_4v_4.$$

De-Sitter space is defined by $X := \left\{x \in \mathbb{R}^4 \mid g(x,x) = -1\right\}$. Show that $X$ is a manifold. Show that the restriction of $g$ to $X$ has signature $(2,1)$. Calculate the shape operator of $X$. Calculate the sectional curvature of $X$.

*N.B. Special care should be taken with the sign, since the metric is not positive definite.*

**Question C:** Consider the conformal metric, $g$, defined over $\mathbb{R}^2$ by

$$g_{ij} = e^{2f}\delta_{ij},$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Determine a formula for the scalar curvature of $g$.

**Question D:** Let $X$ be a riemannian manifold of sectional curvature bounded above by $-1$. Let $p$ be a point of $X$, and let $\phi : X \to \mathbb{R}$ be the squared distance in $X$ to $p$. That is

$$\phi(q) = d(p,q)^2.$$

Show that $\mathrm{Hess}(\phi) \geq 2\mathrm{Id}$.

**Bonus:** Let $\Sigma$ be an embedded hypersurface in $X$ with shape operator, $A$. Calculate, $\Delta^\Sigma\phi$, the laplacian of the restriction of $\phi$ to $\Sigma$. Recall that the *mean curvature* of $\Sigma$ is the arithmetic mean of its principle curvatures. That is,

$$H = \frac{1}{d}\mathrm{Tr}(A),$$

where $d$ is the dimension of $\Sigma$. Show that if $H < 1$, then $\Delta^\Sigma\phi > 0$. Conclude that $\phi$ is constant over $\Sigma$.