The return sequence of the Bowen–Series map for punctured surfaces

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Abstract
For a non-compact hyperbolic surface \( M \) of finite area, we study a certain Poincaré section for the geodesic flow. The canonical, non-invertible factor of the first return map to this section is shown to be pointwise dual ergodic with return sequence \((a_n)\) given by

\[
a_n = \frac{\pi}{4(Area(M) + 2\pi)} \cdot \frac{n}{\log(n)}.
\]

We use this result to deduce that the section map itself is rationally ergodic, and that the geodesic flow associated to \( M \) is ergodic with respect to the Liouville measure.

1 Introduction and statement of main results

The coding of the directed geodesics on a surface \( M \) of negative curvature by a finite alphabet was introduced by Artin and Morse (see [Ar],[Mor1],[Mor2]). An immediate application of this approach leads to a representation of the geodesic flow on \( M \) as a suspension flow over the two-sided shift (see [Se2]). In this context several problems of interest arise. The first is to determine whether the shift is of finite type, which can be done e.g. by giving a geometric construction of an invertible Markov map such that the geodesic flow is isomorphic to a suspension over the latter map (see [Se2],[AF]). Furthermore, one may be interested in the maximal non-invertible factor of this map, its dynamical properties, and their relation to the dynamics of the geodesic flow on the given surface.

A fundamental paper in this context is [BS]. In there, a non-invertible Markov map \( T : \partial \mathbb{H} \to \partial \mathbb{H} \) is introduced which is orbit equivalent to the action of the Fuchsian group \( G \) on \( \mathbb{H} \) where \( \partial \mathbb{H} \) refers to the ideal boundary of the hyperbolic 2-space \( \mathbb{H} \) and \( \mathbb{H}/G \) is assumed to have finite hyperbolic area. If \( \mathbb{H}/G \) is compact, it is shown that the map \( T \) is transitive and satisfies a Rényi property which then implies that \( T \) admits an ergodic invariant probability measure which is absolutely continuous with respect to the Lebesgue measure. If \( \mathbb{H}/G \) is not compact, they prove that a suitably chosen induced transformation has these properties which gives that \( T \) itself is ergodic with respect to an infinite invariant measure. Furthermore, the geodesic flow is shown to be ergodic by using the orbit equivalence of the action of \( T \) and \( G \). Later, it was shown in [Se2] that the map \( T \) also has the above mentioned property of being a maximal factor of the first return map to some Poincaré section for the geodesic flow on \( \mathbb{H}/G \). Note that this relation gives an explicit construction of an invariant measure for \( T \) using the flow invariant Liouville measure on the unit tangent bundle of \( \mathbb{H}/G \). Namely, since the Liouville measure induces an invariant measure for the first return map (see [AK]), the image measure under the factor map of the latter measure is \( T \)-invariant. By using this construction, this invariant measure was determined explicitly in [Se1] for the modular group and in [AF] for compact \( \mathbb{H}/G \) in order to obtain a new characterisation of the Gauss measure, and to prove the ergodicity of the geodesic flow, respectively.
In the sequel, we refer to the map $T$ as the Bowen–Series map or the coding map, and a Fuchsian group $G$ is called cocompact or cofinite if $\mathbb{H}/G$ is compact or of finite hyperbolic area.

In here, we first consider the above construction for an arbitrary, cofinite, non-cocompact Fuchsian group $G$. In this situation, a result of Tukia (see [Tu]) gives that there exists a fundamental domain for $G$ which is an ideal polygon. It then turns out that the $T$–invariant measure is infinite (see [BS], [Se1]). Therefore, we use methods from infinite ergodic theory (see [ADU], [Aa]) to show that the coding map is pointwise dual ergodic and to determine the associated return sequence (see [AD]). After that, a brief adaption of the methods used here is given for the coding map as introduced in [BS]. Note that this construction applies for any cofinite group.

In order to be more precise, the paper is organised as follows. In Section 2, we construct a Poincaré section $Y$ for the geodesic flow on $\mathbb{H}/G$ where $G$ is non-compact and cofinite. Note that the construction of $Y$ relies on the choice of a fundamental polygon (see [Se1], [AF], [St]) which can be chosen to be an ideal polygon (see [Tu]). This then gives rise to a special flow representation (Proposition 2.1) of the geodesic flow. Therefore, the Liouville measure induces a measure $m$ on $Y$ which is invariant under the first return map $S$ to $Y$.

In Section 3, the coding map $T$ is introduced and is shown to be a non-invertible factor of $S$. That is, there is a surjective map $\pi : Y \to \partial \mathbb{H}$ such that $\pi \circ S = \pi \circ T$. Since $m$ is $S$–invariant, the measure $\mu := m \circ \pi^{-1}$ is a $T$–invariant measure. Moreover, this measure can be calculated explicitly (see p. 8). Also, we conclude that $T$ is a topologically mixing Markov map (Proposition 3.1) and that $S$ is the natural extension of $T$ (Proposition 3.2). Combining this result with Corollary 2.2) then gives, that the map $T$ is conservative and ergodic if and only if the geodesic flow has these properties.

In Section 4, we first show that the measure $\mu$ is infinite (Proposition 4.1). Furthermore, by inducing $T$ to a suitable set $A$ of finite measure we deduce that $T_A$ is an eventually hyperbolic dynamical system which satisfies a Rényi property (Lemma 4.3). Using this result, we obtain the following result, where $\mu_A$ refers to the measure $\mu$ restricted to $A$.

**Theorem 1.** The induced map $T_A$ has the Gibbs–Markov property with respect to the $T_A$–invariant measure $\mu_A$.

Note that the Gibbs–Markov property was introduced in [AD] where a similar result was obtained for a subgroup of the modular group by a different method. Moreover, recall that the continued fraction map is the classical example for a map with this property. By combining results from infinite ergodic theory (see [ADU], [Aa], [Z]) and the measure estimate in Proposition 4.1 we obtain the main result of this paper.

**Theorem 2.** The coding map $T$ is conservative and ergodic with respect to $\mu$. Moreover, for $K := \pi/(4(Area(M) + 2\pi))$, the map $T$ is pointwise dual ergodic with respect to the return sequence

$$a_n = K \cdot \frac{n}{\log(n)}.$$

The reader might like to recall that $T$ is called pointwise dual ergodic (see also Section 4.3) with respect to the return sequence $(a_n)$ if, where $\hat{T}$ refers to the dual of $T$,

$$\frac{1}{a_n} \sum_{i=0}^{n-1} \hat{T}^i f \to \int_X f d\mu \text{ a.e. as } n \to \infty \forall f \in L^1(\mu).$$

Note that the sequence $(a_n)$ is unique up to asymptotic equality. That is, the $T$ is pointwise dual ergodic with respect to each sequence $(b_n)$ with $\lim_{n \to \infty} a_n/b_n = 1$. Moreover, the measure $\mu$ can be normalised in the following sense. With respect to the measure $\mu$,
(1/K) · µ, the map T is pointwise dual ergodic with return sequence (n/ log n). However, this normalisation corresponds to a change of the Riemannian metric on \( \mathbb{H} \) and therefore to a change of the sectional curvature of \( \mathbb{H}/G \).

In Section 5, the relation of the coding map which is constructed in Section 3 and the map T introduced in [BS] is discussed. It turns out that the analysis in here can be adapted to this more general situation as follows. Using the above mentioned construction of the invariant measure, a slight modification of Lemma 4.3 then gives the following result, where \( V^* \) refers to the set of ideal vertices of the polygon involved in the construction of T.

**Theorem 3 & 4.** Let \( G \) be a cofinite group and \( T \) be the map as introduced in [BS]. We then have that the map \( T \) is conservative and ergodic with respect to the invariant measure \( \mu \). Moreover, the measure \( \mu \) is finite if and only if \( G \) is cocompact. If \( G \) is not cocompact, the map \( T \) is pointwise dual ergodic with respect to the return sequence

\[ a_n = K \cdot \frac{n}{\log n}, \]

where \( K = 1/(4 \# V^*) \).

Note that the latter result comprises the assertion of Theorem 2. This is due to the fact, that for an ideal polygon \( P \) with \( n \) vertices, the hyperbolic area of \( P \) is given by \( \text{Area}(P) = \pi(n - 2) \).

These results have the following consequences. Since the natural extension of \( T \) is the first return map to a Poincaré section (see Proposition 3.2 for the construction given here and [Se2],[AF] for the construction in [BS]), the geodesic flow is isomorphic to a special flow over a conservative and ergodic section (Proposition 2.1). Hence the classical result of Hopf (see e.g. [Ho]) which states that the geodesic flow on a surface of finite area is conservative and ergodic with respect to the Liouville measure follows by a result in [AK]. Furthermore, if the invariant measure for this section is infinite, the fact that the first return map is conservative implies that \( S_A \) is well defined for each subset \( A \) of positive measure of the Poincaré section. Hence for a suitably chosen set \( A \), the induced map \( S_A : A \to A \) is the first return map to the alternative Poincaré section \( A \), and is an ergodic, finite measure preserving Markov map with respect to a countable partition (see [AD]).

2 The special flow representation

The aim of this section is to construct a special flow over some measure preserving transformation which is isomorphic to the geodesic flow on the underlying surface. Therefore recall that the *Poincaré model* \( \mathbb{H} \) of the hyperbolic plane is given by \( \mathbb{H} := \{ z \in \mathbb{C} \mid |z| < 1 \} \) where the *hyperbolic metric* and *hyperbolic area* are given by

\[ ds(z) = \frac{2|dz|}{1 - |z|^2} \quad \text{and} \quad dA(z) = \frac{4dz}{(1 - |z|^2)^2}, \]

respectively. Also recall that an oriented *geodesic* is an isometry \( \gamma : \mathbb{R} \to \mathbb{H} \) and corresponds to a circle segment which is perpendicular to the boundary at infinity \( \partial \mathbb{H} = S^1 \) at the two endpoints \( \lim_{t \to \pm \infty} \gamma(t) \). Note that the set of oriented geodesics corresponds to the unit tangent bundle \( T^1(\mathbb{H}) \) of \( \mathbb{H} \) where a common representation of \( T^1(\mathbb{H}) \) is given by

\[ T^1(\mathbb{H}) = \{ (\xi, \eta, s) : \xi, \eta \in \partial \mathbb{H}, \xi \neq \eta, s \in \mathbb{R} \}. \]

Moreover, the geodesic flow \( (\phi_t)_{t \in \mathbb{R}} \) on \( T^1(\mathbb{H}) \) is defined by the canonical \( \mathbb{R} \)-action \( \phi_t : (\xi, \eta, s) \mapsto (\xi, \eta, s + t) \), and the *Liouville measure* \( m_L \) is invariant under the action of \( (\phi_t)_{t \in \mathbb{R}} \), where \( m_L \) is given by \( dm_L = d|\xi||\eta|dt/||\xi - \eta||^2 \).
Recall that each orientable hyperbolic surface is isometric to the quotient $\mathbb{H}/G$ where $G$ is Fuchsian group without torsion. Namely, a group $G$ is torsion-free if there exists no non-trivial element $g \in G$ such that $g^n = id$ for some $n \neq 0$. Furthermore, a group is called a Fuchsian group if $G$ is a discrete subgroup of the group of orientation-preserving isometries $\text{Iso}^+(\mathbb{H})$ of $(\mathbb{H}, s)$. Also note that $m_L$ is invariant under the common action of $G$ on $T^1(\mathbb{H})$, and that the actions of $G$ and $(\phi_t)_{t \in \mathbb{R}}$ on $T^1(\mathbb{H})$ commute. These observations give rise to the definition of the geodesic flow on the quotient $T^1(\mathbb{H}/G)$ with respect to the projected Liouville measure which, for ease of notation, will also be denoted by $m_L$ (see [Ni]).

In the sequel we will extensively use the choice of a specially shaped fundamental polygon for a given group $G$. Recall that in general, the set of sides $S$ of a fundamental polygon $P$ consists of geodesic segments, geodesic rays and geodesics, and that the set of vertices $V$ of $P$ is a subset of $\partial \mathbb{H}$. Also, the elements of $S$ are equivalent in pairs, and for each pair $\{s, t\}$ the element $g_s \in G$ is uniquely determined by $g_s(s) = t$. This gives rise to the following notation. Denote by $s'$ the side $g_s(s)$ of $P$. As it is easily seen, $s'' = s$ and $g_s' = g_s^{-1}$.

So assume that $G$ is a torsion-free, non-cocompact Fuchsian group. In this situation, we are able to apply a result of Tukia ([Tu, p. 15]), which states in this case that there exists a fundamental polygon $P$ for $G$ with the following properties. The sides of the polygon $P$ are geodesics. For each $s \in S$, denote by $G_{\{s, s'\}}$ the subgroup of $G$ generated by the element $g_s$. Then $G$ is the free product of the groups $G_{\{s, s'\}}$ where the product is taken over all pairs $\{s, s'\}$.

If $G$ corresponds to a non-compact surface of finite area, the above result gives that there exists $P$ such that the sides of $P$ are geodesics. Therefore, since $P$ is of finite area, $P$ has to be an ideal polygon. That is $P$ is the (hyperbolically) convex hull of a finite subset $\mathcal{V}$ of $\partial \mathbb{H}$ (see Figure 1). Note that this implies that $\mathcal{S}$ is finite. Moreover, by [Be, Theorem 10.5.1], we have for each fundamental polygon for $G$ that $\#S \leq 4g + 2n - 2$, where $g$ refers to the genus and $n$ to the number of cusps of $\mathbb{H}/G$. Using the Gauß-Bonnet formula (e.g. [Be, Theorem 10.4.3]) it is not hard to deduce that this inequality is sharp for each ideal fundamental polygon. In other words, this choice of $P$ has the fewest number of sides and therefore gives a minimal set of generators.

In addition, since $\mathcal{V} \subset \partial \mathbb{H}$, the polygon $P$ gives rise to a partition of $\partial \mathbb{H}$ as follows (see Figure 1). Denote by $H(s)$, for $s \in \mathcal{S}$, the open hyperbolic half-space for which $\partial H(s) = s$ and $P \cap H(s) = \emptyset$ and by $a_s \subset \partial \mathbb{H}$ the open interval which is adjacent to $H(s)$. As it is easily seen, $g_s(a_s) = \text{Int}(a_s')$ where $\text{Int}(\cdot)$ refers to the interior with respect to the topology of $\partial \mathbb{H}$. Furthermore, $a_s \cap a_t = \emptyset$ for all distinct $s, t \in \mathcal{S}$ and $\bigcup_{s \in \mathcal{S}} a_s = \partial \mathbb{H} \setminus \mathcal{V}$.
Consider the following subset $\mathcal{G}_P$ of $T^1(\mathbb{H})$ which is given by

$$\mathcal{G}_P := \{ (\xi, \eta, t) \in T^1\mathbb{H} : \xi \in GV \text{ or } \eta \in GV \}.$$ 

Clearly, $\mathcal{G}_P$ is invariant under the action of $(\phi_t)$, and under the action of $G$. Moreover, since $\mathcal{V}$ is finite, $\mathcal{G}_P$ is a set of Liouville measure zero. Let $\bar{\gamma}_{\xi, \eta}$ refer to the oriented geodesic from $\eta \in \partial \mathbb{H}$ to $\xi \in \partial \mathbb{H}$ where $\bar{\gamma}_{\xi, \eta}$ is normalised such that the Euclidean distances $d_E(\eta, \gamma(0))$ and $d_E(\xi, \gamma(0))$ coincide. We now define the set

$$Y := \{ (\xi, \eta) \in \partial \mathbb{H} \times \partial \mathbb{H} : \exists t \in \mathbb{R} \text{ such that } (\xi, \eta, t) \notin \mathcal{G}_P, \text{ and } \bar{\gamma}_{\xi, \eta}(t) \in P \}.$$ 

Furthermore, for distinct $\xi, \eta \in \partial \mathbb{H} \setminus \mathcal{V}$, observe that there exists $t \in \mathbb{R}$ such that $\bar{\gamma}_{\xi, \eta}(t) \in \mathrm{Int}(P)$ if and only if there exists distinct $s, t \in S$ such that $\xi \in a_s$ and $\eta \in a_t$. Thus,

$$Y \overset{m}{=} \{ (\xi, \eta) \in \partial \mathbb{H} \times \partial \mathbb{H} : \exists s \in S \text{ such that } \xi \in a_s, \eta \notin a_s \},$$

where $m$ is given by $dm(\xi, \eta) = d[|d|\eta/|\xi - \eta|^2]$ and $\overset{m}{=} \text{ denotes equality up to a set of measure zero.}$

Let

$$S : Y \to Y, \quad S|_{(a_s \times a_t)}(\xi, \eta) = (g_s \xi, g_t \eta).$$

Moreover, since we have excluded the set $\mathcal{G}_P$, observe that for the two maps $t_{\xi, \eta}^+: Y \to \mathbb{R}$, defined by

$$t_{\xi, \eta}^+ := \sup \{ t \mid \gamma_{\xi, \eta}(t) \in P \} \leq \infty,$$

$$t_{\xi, \eta}^- := \inf \{ t \mid \gamma_{\xi, \eta}(t) \in P \} \geq -\infty,$$

we have that $|t_{\xi, \eta}^+| < \infty$ for all $(\xi, \eta) \in Y$.

Recall that $(\text{see } [AK])$ the special flow $(Y_h, B_h, m \times \lambda, (\varphi_{t_h}^Y))_{t \in \mathbb{R}}$ over $S : (Y, \mathcal{B}, m) \to (Y, \mathcal{B}, m)$ with height function $h(\xi, \eta) := t_{\xi, \eta}^+ - t_{\xi, \eta}^-$ is defined by,

$$Y_h := \{ (\xi, \eta, \theta) \mid (\xi, \eta) \in Y, 0 \leq \theta < h((\xi, \eta)) \},$$

$$\varphi_{t_h}^Y(\xi, \eta, \theta) := (S^n(\xi, \eta), \theta + t - h_n(x)) \text{ where } n \in \mathbb{Z} \text{ is given by }$$

$$h_n(\xi, \eta) := \begin{cases} 0 & : n = 0 \\ \sum_{k=0}^{n-1} h(T^{k}(\xi, \eta)) & : n \geq 1 \\ -\sum_{k=n}^{0} h_k(T^{k}(\xi, \eta)) & : n < 0. \end{cases}$$

In here, $m \times \lambda$ refers to the product measure of $m$ and the Lebesgue measure $\lambda$ restricted to the Borel $\sigma$-field $B_h$ of $Y_h$.

In this context, $S$ is also referred to as the first return map to the Poincaré section $Y$. Note that, by results of $[AK]$, that the measure $m$ is $S$-invariant if and only if $m \times \lambda$ is invariant under $(\varphi_{t_h}^Y)_{t \in \mathbb{R}}$, and that $S$ is ergodic and conservative if and only if $(\varphi_{t_h}^Y)_{t \in \mathbb{R}}$ is ergodic and conservative.

**Proposition 2.1.** The geodesic flow $(T^1\mathbb{H} \mid \mathcal{G}_P, \mathcal{B}, m_L, (\phi_t))$ is measure theoretically isomorphic to the special flow $(Y_h, B_h, m \times \lambda, (\varphi_{t_h}^Y))$.

**Proof:** In here, we only give the sketch of this proof since similar arguments can be found in $[Se1]$ and $[AF]$. As it is easily seen, the set $\mathcal{Y}$ is a fundamental domain for the action of $G$ on $T^1(\mathbb{H}) \setminus \mathcal{G}_P$, where $\mathcal{Y}$ is given by

$$\mathcal{Y} := \{ (\xi, \eta, t) \in Y \times \mathbb{R} : t_{\xi, \eta}^- \leq \theta < t_{\xi, \eta}^+ \}.$$ 

In addition, for $g \in G$ we have that $g\mathcal{Y} \cap \mathcal{Y} = \emptyset$ if and only if $g = id$. Hence by the product structure of $m_L$ and the flow invariance of $\mathcal{G}_P$ we obtain that the geodesic
flow \((\phi_t)\) on \(T^1\HH/G\) with respect to the Liouville measure and the flow \((\psi_t)\) on \(\YY\) with respect to \(m \times \lambda\) are measure theoretically isomorphic, where \((\psi_t)\) is given by 
\[
\psi_t(\zeta, \eta, \theta) = G(\phi_t(\zeta, \eta, \theta)) \cap \YY.
\]

Moreover, observe that for \(\xi \in a_s\), we have that 
\[
g_s(\xi, \eta, t_{\xi, \eta}^+ t_{\xi, \eta}^-) = (g_s(\xi, \eta, t_{g_s(\xi, \eta)})).
\]
This essentially gives the assertion. \(\square\)

Note that \(h \in L^1(Y, m)\), since we have that 
\[
\int_Y hdm = m\left(T^1(\HH/G)\right) = \text{Area}(\HH/G).
\]

Furthermore, since the Liouville measure is flow–invariant we immediately obtain the following.

**Corollary 2.2.** The map \(S : Y \to Y\) is the first return map of the Poincaré section \(Y\). Moreover, the measure \(m\) is \(S\)–invariant and \(S\) is conservative and ergodic if and only if the geodesic flow is conservative and ergodic.

### 3 The coding map

The coding map is an endomorphism defined on \(\partial \HH\) which is defined piecewise by a set of generators given by a fundamental polygon. Note that 
\[
\alpha := \{a_s : s \in \SS\}
\]
is a partition of \(\partial \HH\) up to a set of Lebesgue measure zero. The map \(T : \partial \HH \to \partial \HH\) is now defined, where \(a_s\) refers to an arbitrary atom of \(\alpha\), by 
\[
T|_{a_s} := g_s|_{a_s}.
\]

Observe that, where \(pr_1\) refers to the projection onto the first coordinate, that 
\[
pr_1 \circ S = T \circ pr_1.
\]
Hence \(T\) is a factor of \(S\) and the measure \(\mu := m \circ pr_1^{-1}\) is \(T\)–invariant. Moreover, for a countable collection of partitions \(\{\beta_i : i \in I\}\), denote by \(\bigvee_{i \in I} \beta_i\) the common refinement of the \(\beta_i\) \((i \in I)\). Let 
\[
\alpha_{n+1} := \bigvee_{i=0}^{n} T^{-i}\alpha.
\]

**Proposition 3.1.** The coding map \(T\) is a topologically mixing Markov map with respect to the partition \(\alpha\) and the measure \(\mu\) (and with respect to the Lebesgue measure).

**Proof:** \(T\) restricted to an element of \(\alpha\) is a Möbius transformation whence \(T|_{a_s}\) is injective. Moreover, since \(T|_{a_s} = g_s|_{a_s}\), we have that \(T(a_s) = (a_s')^e\) mod \(\mu\). To verify the Markov property it remains to show that 
\[
\sigma(\bigvee_{i=0}^{\infty} T^{-i}\alpha) = B\mod\mu.
\]
As it is easily seen, the inverse branches of \(T\) correspond to elements of the group \(G\). Hence any element of \(\alpha_n\) is corresponding to a side of a copy of \(P\) by an element of \(G\). Note that, since the tessellation \(GP\) is locally finite, the Euclidean distances of the endpoints of the sides of \(g_nP\) tend to zero where \(g_n\) is a sequence of mutually distinct elements in \(G\). This gives the Markov property with respect to \(\alpha\).

In order to complete the proof, it remains to show that \(T\) is topologically mixing which is equivalent to the aperiodicity of the underlying incidence graph (see [Aa, §4.2]). Recall that the set of vertices of this graph consists of the elements of \(\alpha\), and that the set of (directed) edges consists of the pairs \((a, b)\) with the property that 
\[
T(a) \supset b \mod \mu.
\]
Since \(T(a) \supset b\) is equivalent to \(b \neq a'\) there are edge cycles \(((a_0, a_1), (a_1, a_2), \ldots)\).
\((a_{k-1}, a_k), (a_k, a_0)\) of arbitrary length for all \(a = a_0\). Hence the incidence graph is aperiodic and \(T\) is topologically mixing.

The Markov property now allows to introduce the following notions. A word \((s_1 \ldots s_n)\) is called admissible whenever \(s_i \neq s'_{i+1}\) for \(1 \leq i < n\) (which is equivalent to \(T(a_{s_i}) \supset a_{s_{i+1}}\)). Note that each admissible word \(\omega = (s_1 \ldots s_n)\) defines an element \(\lfloor \omega \rfloor\) of \(\alpha_n\) by

\[\lfloor \omega \rfloor := \{\xi \in \partial \mathbb{H} : T^i(\xi) \in a_{s_i} \text{ for all } 0 \leq i < n - 1\}.
\]

Moreover, if \(g_\omega \in G\) refers to \(g_\omega = g_{s_n} \cdots g_{s_1}\), then \(T^n|\lfloor \omega \rfloor : \omega \to T([a_{s_1}])\) is injective and \(T^n|\lfloor \omega \rfloor = g_\omega\). This gives the following well known relation between admissible words and inverse branches of \(T\). Namely, for \(\mathcal{D}(\nu_\omega) := T^n|\lfloor \omega \rfloor\) we have that

\[\nu_\omega : \mathcal{D}(\nu_\omega) \to \lfloor \omega \rfloor, \nu_\omega := g_\omega^{-1}|T^n(\lfloor \omega \rfloor) \text{ and } T^n \circ \nu_\omega|\mathcal{D}(\nu_\omega) = \text{id}|\mathcal{D}(\nu_\omega).
\]

We are now in position to show that the first return map \(S\) also has the Markov property. Therefore recall that \(S\) is referred to as the natural extension of \(T\) if \(\text{pr}_1 \circ S = T \circ \text{pr}_1, m \circ \pi^{-1} = \mu\), and \(\bigwedge_{n=1}^\infty S^n \text{pr}_1^{-1} B_{\partial \mathbb{H}} \equiv B_Y\) where \(B_{\partial \mathbb{H}}\) resp. \(B_Y\) denote the \(\sigma\)-field of Borel sets of \(\partial \mathbb{H}\) resp. \(Y\) (see e.g. [An]).

**Proposition 3.2.** \((Y, B_Y, m, S)\) is the natural extension of \((\partial \mathbb{H}, B_{\partial \mathbb{H}}, \mu, T)\).

**Proof:** Assume that \((s_1 \ldots s_m)\) is an admissible word and that \(0 < n < m\). Then

\[S^n \circ \text{pr}_1^{-1}[s_1 \ldots s_m] = g_{s_n} \cdots g_{s_1}(s_1 \ldots s_m) \times g_{s_n} \cdots g_{s_1}([s_1]^-) = [s_{n+1} \ldots s_m] \times (g_{s'_n} \cdots g_{s'_1})^{-1} T([s'_1]) = [s_{n+1} \ldots s_m] \times [s'_1 \ldots s'_n].
\]

As \(\alpha\) is a generating partition with respect to \(T\) the last equality proves that the Borel sets of \(Y\) are generated by

\[\bigwedge_{m > n > 0} S^n \circ \text{pr}_1^{-1}(\alpha_m).
\]

By definition of \(T\) and \(\mu\) the two other criteria are fulfilled and hence the assertion is proven.

The following corollary is an immediate consequence of the latter proposition.

**Corollary 3.3.** The first return map \(S\) has the Markov property with respect to the partition \(\{a_s \times a_t : s, t \in S, s \neq t\}\) of \(Y\).

## 4 Ergodic properties of the coding map

By Proposition 3.1 the map \(T\) is a measure preserving Markov map with respect to a partition with finitely many atoms. However, the fact that \(\mathbb{H}/G\) is a surface with cusps implies that there exist indifferent periodic orbits of \(T\). This then will give rise to the observation that \(\mu\) is an infinite measure (see [BS],[Sel]), and therefore we will describe the dynamical behaviour of \(T\) in terms of infinite ergodic theory (see [Th],[ADU],[An]).

Denote by \(U := \{z \in \mathbb{C} : \Im z > 0\}\) the upper half space model of the hyperbolic plane. Recall that the Liouville measure \(m_L\) on \(T^1U = \{(\xi, \eta, t) : \xi, \eta \in \mathbb{R} \cup \{\infty\}, \xi \neq \eta, t \in \mathbb{R}\}\) is given by

\[dn_L = \frac{2d\xi d\eta ds}{(\xi - \eta)^2}.\]
Hence, the invariant measure \( m \) for the first return map \( S \) with respect to \( U \) is given by

\[
dm = \frac{2d\xi d\eta}{(\xi - \eta)^2},
\]

and as in the Poincaré model, the invariant measure \( \mu \) for \( T \) is the image measure of the projection onto the first coordinate. We now consider an arbitrary atom \( a_s \) of \( \alpha \) and assume without loss of generality that \( a_s = (a, \infty) \) for some \( a \in \mathbb{R} \). Hence, for each measurable set \( A \subset a_s \), we have

\[
(1) \quad \mu(A \times a_s) = \int_{A \times a_s} \frac{2d\xi d\eta}{(\xi - \eta)^2} = \int_A \left( \int_{a_s} \frac{2d\eta}{(\xi - \eta)^2} \right) d\xi.
\]

Hence,

\[
(2) \quad \frac{d\mu}{d\xi}(\xi) = \int_{-\infty}^{\alpha} \frac{2d\eta}{(\xi - \eta)^2} = \frac{2}{\eta - a} d\eta \quad \text{for all } \xi \in a_s,
\]

and \( \mu \) is an infinite measure which is equivalent to the Lebesgue measure on \( \mathbb{R} \).

### 4.1 The wandering rate

The wandering rate of a set \( A \) of finite measure with respect to a measure preserving transformation is given by the asymptotic type of \( \mu(\bigcup_{i=1}^{n} T^{-i}(A)) \) where \( n \to \infty \). Note that \( T \) is conservative if \( \mu(\partial U \setminus \bigcup_{i=1}^{\infty} T^{-i}(A)) = 0 \). To construct a set with the latter property the indifferent periodic orbits of \( T \) have to be characterised. Recall the definition of a cycle for an ideal vertex of a fundamental polygon \( P \) for the group \( G \). Assume that, for the ideal vertices \( v_1, \ldots, v_n \in \partial \mathbb{H} \), for the sides \( s_1, \ldots, s_n \in \mathcal{S} \), and for the boundary identifications \( g_{s_1}, \ldots, g_{s_n} \), the following holds, where the indices are considered to be taken mod \( n \),

- \( g_s(v_i) = v_{i+1} \) for \( 0 < i \leq n \),
- \( v_i \) is adjacent to \( s_i \) and \( v_{i+1} \) is adjacent to \( g_s(s_i) \) for all \( 0 < i \leq n \).

If \( n \) is minimal with respect to these properties, then \( (v_1, \ldots, v_n) \) is referred to as a vertex cycle of \( v_1 \). For the case of a cofinite group, it is well known that each ideal vertex \( v \) of \( P \) is contained in a vertex cycle. Moreover, for the associated boundary identifications \( g_{s_1}, \ldots, g_{s_n} \), we have that \( g_{s_{n}}, g_{s_{n-1}}, \ldots, g_{s_1} \) is a parabolic transformation with fixed point \( v \).

Let \( N \) refer to the smallest common multiple of the lengths of all vertex cycles. Hence each ideal vertex \( v \) is a parabolic fixed point of \( T^N \) where \( T^N|_{[w]} \) refers to the continuous extension of \( T^N \) to \([w] \in \alpha N \). As it is easily seen, where \((s_1, \ldots, s_n)\) refers to the cycle of sides associated to \( v \), that

\[
U(v) := [s_1 \ldots s_n s_1 \ldots s_n \ldots s_1 \ldots s_n] \cup [s'_{n} \ldots s'_{1} s'_{1} \ldots s'_{n}] \cup \{v\}
\]

is a neighbourhood of \( v \). Denote by

\[
w(v) := [s_1 \ldots s_n s_1 \ldots s_n \ldots s_1 \ldots s_n],
\]

\[
w'(v) := [s'_{n} \ldots s'_{1} s'_{1} \ldots s'_{n}].
\]
Since $w(v)w(v)$ is admissible, we have that

$$
\bigcup_{i=1}^{n} T^{-iN}([w(v)])^c = \underbrace{[w(v) \ldots w(v)]}_{n+1 \text{ times}}^c.
$$

For $A := \left( \bigcup_{v \in V} U(v) \right)^c$ this gives that

$$
\bigcup_{i=0}^{n} T^{-iN} A \overset{\mu}{=} \bigcup_{i=0}^{n} T^{-iN} \left( \bigcup_{v \in V} \left( [w(v)] \cup [w'(v)] \right) \right)^c
$$

$$
= \left( \bigcup_{v \in V} \underbrace{[w(v) \ldots w(v)]}_{n+1 \text{ times}} \cup \underbrace{[w'(v) \ldots w'(v)]}_{n+1 \text{ times}} \right)^c.
$$

Without loss of generality, assume that $v = \infty$, and that $T^N[w(v)](z) = z - 1$. Hence there exist $a, b \in \mathbb{R}$, $b < a$ such that $w(v) = (a, \infty)$, and that $(b, \infty) = [s]$ where $[s] \in \alpha$ is the atom in $\alpha$ such that $w(v) \subset [s]$. We now have that

$$
\mu([w(v)]|\underbrace{[w(v) \ldots w(v)]}_{\text{n times}}) = \mu((a, a + n])
$$

$$
= \int_{a}^{a+n} \frac{2}{x-b} dx = 2 \left( \log(a+n-b) - \log(a-b) \right)
$$

$$
\Rightarrow \mu([w(v)]|\underbrace{[w(v) \ldots w(v)]}_{\text{n times}}) \xrightarrow{\text{n} \to \infty} 2.
$$

This observation now implies that $\mu(A) < \infty$, and that for the wandering rate of $A$ with respect to $T^N$, where $\#V$ denotes the cardinality of $V$, we have

$$
\frac{\mu(\bigcup_{i=0}^{n} T^{-iN} A)}{\log n} \xrightarrow{n \to \infty} 4\#V.
$$

Since $\mu(\bigcup_{i=0}^{n} T^{-iA})_{i \in \mathbb{N}}$ increases monotonically, we have the following.

**Proposition 4.1.**

$$
\frac{\mu(\bigcup_{i=0}^{n} T^{-iA})}{\log n} \xrightarrow{n \to \infty} 4 \#V.
$$

In addition, since $(a, a + n] \xrightarrow{n \to \infty} (a, \infty)$, we have that $\mu(\partial U \setminus \bigcup_{i=1}^{\infty} T^{-i}(A)) = 0$.

**Proposition 4.2.** The first return map $T_A : A \to A$ is well defined and preserves the finite measure $\mu$ restricted to $A$. Furthermore $T_A$ and $T$ are conservative.

**Proof:** As $\mu(\partial U \setminus \bigcup_{i=1}^{\infty} T^{-i}(A)) = 0$, $T_A$ is defined almost everywhere. As $\mu(A)$ is finite, $T_A$ preserves a finite measure and hence is conservative. By standard arguments $T$ has to be conservative as well.

### 4.2 Distortion properties

In order to derive distortion properties like the Rényi or Gibbs property for the induced transformation $T_B$ for some measurable set $B$ of finite measure, recall the following. For each Möbius transformation $g$ which does not fix $\infty$ there exists a unique circle $I_g$, the isometric circle, on which $g$ acts as an Euclidean isometry. If $g$ in particular is an isometry of the Poincaré model and $g(0) \neq 0$ then there exist a reflection $\tau$ at a straight
line through the origin such that $g = \tau \sigma$ where $\sigma$ is the inversion in $I_g$. In addition, $I_g$ is perpendicular to $S^1$ and hence corresponds to a geodesic (see [Ra, §4.3]).

Denote by $m_g$ the centre and by $r_g$ the radius of $I_g$. Since $I_g$ is perpendicular to $S^1$, we have that $|m_g|^2 = r_g^2 + 1$. The inversion $\sigma$ in $I_g$ is given by

$$\sigma(z) = \frac{m_g \bar{z} - |m_g|^2 + r_g^2}{\bar{z} - m_g} = \frac{m_g \bar{z} - 1}{\bar{z} - m_g}.$$ 

Hence, where $D(\cdot)$ refers to the derivative of a holomorphic function, we have that

$$|Dg(z)| = \left| \frac{r_g^2}{|z - m_g|^2} \right| \quad \text{and} \quad |D^2g(z)| = \left| \frac{2r_g^2}{(z - m_g)^3} \right|.$$ 

If $\psi_g$ refers to the repelling (indifferent) fixed point of the hyperbolic (parabolic) transformation $g$ then $|g'(z)| \leq 1$ if and only if $|z - m_g| \leq r_g$. Thus $|\psi_g - m_g| \leq r_g$. Furthermore, we have that

$$\left| \frac{D^2g(z)}{(Dg(z))^2} \right| = \frac{2|z - m_g|}{r_g^2}.$$ 

Recall that any Moebius transformation leaves invariant the cross ratio $[u,v,x,y]$ where $u, v, x, y$ are four different points in $\mathbb{C} \cup \{\infty\}$ and the cross ratio is given by

$$[u,v,x,y] := \frac{|u - x||u - v|}{|u - v||x - y|}.$$ 

**Lemma 4.3.** With respect to the disc model, we have that, for each $B$ measurable with the property that $d(B,V) > \epsilon$ for some $\epsilon > 0$ (e.g. $B = A$ as in Proposition 4.2), there exists $0 < C < \infty$ such that, for all $n \in \mathbb{N}$ and for Lebesgue a.e. $z$ with $T^n(z) \in B$,

$$\left| \frac{D^2T^n(z)}{(DT^n(z))^2} \right| < C.$$ 

**Proof:** Fix $\omega = (s_1 \ldots s_n) \in \alpha^n$. Then $T^n[\omega] = g_\omega = g$. Assume that $\eta_g$ is an element of the isometric circle $I(g)$ of $g$ with center $m_g$ and radius $r_g$. Since $g(I_g) = I_{g^{-1}}$ it follows that $r_g = r_{g^{-1}}$ and $m_{g^{-1}} = g(\infty)$. Hence

$$[m_g, \eta_g, z, \infty] = [g(m_g), g(\eta_g), g(z), g(\infty)]$$

$$\Rightarrow \quad \frac{|m_g - z|}{|m_g - \eta_g|} = \frac{|m_{g^{-1}} - g(\eta_g)|}{|m_{g^{-1}} - g(z)|}$$

$$\Rightarrow \quad \frac{|m_g - z|}{r_g} = \frac{|m_{g^{-1}} - g(z)|}{r_g}.$$ 

Therefore, by equation (3),

$$\left| \frac{D^2g(z)}{(Dg(z))^2} \right| = \frac{2}{|m_{g^{-1}} - g(z)|}.$$ 

Hence it remains to derive an estimate of $|m_{g^{-1}} - g(z)|$ from below for all $\omega$ and $z$ with $z \in B \cup [\omega]$ and $T^n(z) \in B$.

**Case 1:** Assume that $s_1 \neq s'_n$. Then $[\omega] \subset T^n[\omega] = g([\omega]) \notin [s'_n]$. Fix $z \in [\omega] \cap B$ with $g(z) \in B$. By the intermediate value theorem for continuous functions it follows that $\text{Clos}([\omega])$ contains a point which is fixed by $g$. By the same argument $\text{Clos}([s'_n]) = g([\omega])$ contains a fixed point of $g^{-1}$. Clearly, the latter fixed point is a repelling or indifferent
fixed point of $g^{-1}$. Hence $\psi_{g^{-1}} \in \text{Clos}(s_n')$. Since $g(z) \in (a_{\omega'})^c \cap B$ and $d(B, V) > \epsilon$, the inequality $|\psi_{g^{-1}} - m_{g^{-1}}| \leq r_{g^{-1}}$ implies that $|m_{g^{-1}} - g(z)| \geq \epsilon - r_{g^{-1}}$. Now assume that $(g_1, g_2, \ldots)$ is a sequence of distinct elements of $G$. Then by Theorem 3.3.7 in [Ka], $r_{g_n} \to 0$ as $k \to \infty$. Hence

$$\left| \frac{D^2 g(z)}{(Dg(z))^2} \right| = \frac{2}{|m_{g^{-1}} - g(z)|} \leq \frac{4}{\epsilon}$$

for at most finitely many $g \in G$.

**Case 2:** Assume that $s_1 = s_n'$. In this case $[\omega]$ and $g([\omega])$ are disjoint. Hence neither $[\omega]$ nor $g([\omega])$ contain any fixed point of $g^{-1}$. Hence by the same arguments as above for $z \in B$ with $g(z) \in B$, we have that

$$\left| \frac{D^2 g(z)}{(Dg(z))^2} \right| = \frac{2}{|m_{g^{-1}} - g(z)|} \leq \frac{4}{\epsilon}$$

for at most finitely many $g \in G$.

Recall that the set $A$ of Proposition 4.2 is defined by

$$A = \partial H \setminus \left( \bigcup_{v \in V} U(v) \right) = \partial H \setminus \left( \bigcup_{v \in V} ([w(v)] \cup [w'(v)]) \right)$$

where $N$ is the smallest common multiple of the lengths of the edge cycles. Denote by

$$\bar{\alpha} := \prod_{n=0}^{\infty} \alpha^n.$$ 

We now introduce the following two partitions of $\partial H$.

$$\beta^* := \{ a \in \alpha^N : a \neq [w(v)], a \neq [w'(v)] \forall v \in V \},$$

$$\beta := \{ b \in \bar{\alpha} : \exists a_1, a_2 \in \beta^* : b \subset a_1, T_A(b) = a_2, T_A : b \to a_2 \text{ is injective} \}.$$ 

By definition, $T_A(b) \in \beta^*$ for all $b \in \beta$. Since $\beta^*$ is a finite partition of $A$ there exists a constant $C > 0$ such that $\lambda(T_A(b)) > C \forall b \in \beta$, or in other words, $T_A$ has the big image property with respect to $\beta$ and the Lebesgue measure $\lambda$ restricted to $A$.

**Proposition 4.4.** $(A, B, \lambda, T_A, \beta)$ is a topologically mixing Markov map which is eventually expanding (i.e. there is $\Lambda > 1$ and $n_0 \in \mathbb{N}$ such that $|DT_A^n(z)| > \Lambda$ for all $n > n_0$ and $\lambda$–a.e. $z \in \partial H$). Furthermore, $T_A$ has the Rényi property, i.e. there is $C > 0$ with

$$\left| \frac{D^2 T_A^n(z)}{(DT_A^n(z))^2} \right| < C \forall n \in \mathbb{N} \text{ and for Lebesgue–a.e. } z.$$
Proof: Clearly, $\beta$ is a Markov partition and $T_A$ is topologically mixing. Since the Rényi property follows immediately from Lemma 4.3, it remains to show that $T_A$ is eventually expanding. Moreover, the Rényi property gives rise to an estimate of the diameter of an element $b \in \beta^n$. By a straightforward calculation (which can be found e.g. in [Aa, p. 145]) there is $M > 0$ such that, where $\nu_b$ denotes the inverse branch of $T_A$ on $b \in \beta^n$ for arbitrary $n \in \mathbb{N}$,

$$\exp(-M) \frac{\lambda(b)}{\lambda(T_A(b))} \leq |\nu_b'(z)| \leq \exp(M) \frac{\lambda(b)}{\lambda(T_A(b))}.$$ 

Since $\sup \{\lambda(b) : b \in \beta^n\}$ tends to zero as $n \to \infty$ and since $\beta$ has the big image property, it follows that $T_A$ is eventually expanding.

From a general point of view, a distortion property is a feature of the multiplicative variation of the Radon–Nikodým derivative $\nu'_b := d\mu \circ \nu_b / d\mu$ where $\nu_b : \mathcal{D}(\nu_b) \to [\omega]$ is the inverse branch determined by the admissible word $\omega$. Denote by

$$\tilde{\beta}_+ := \{a \in \tilde{\beta} : \mu(a) > 0\} \quad \text{where} \quad \tilde{\beta} := \bigcup_{n=0}^{\infty} \beta^n.$$

Recall that the Markov map $(X, \mathcal{B}, \mu, T, \beta)$ has the Gibbs property if there is $C > 1$ and $0 < r < 1$ such that $g_r(C, T) = \tilde{\beta}_+$ where $g_r(C, T)$ is given by

$$g_r(C, T) := \{a \in \tilde{\beta}_+ : \left| \log \frac{\nu'_a(x)}{\nu'_a(y)} \right| \leq Cr t(x, y) \text{ for } \mu \times \mu \text{-a.e. } (x, y) \in (\mathcal{D}(\nu_a))^2\}.$$

In here $t : \bigcup_{a,b \in \beta} a \times b \to \mathbb{N} \cup \{0\}$ is defined by

$$t(x, y) := \min\{n \geq 0 : T^n x \in a, T^n y \in b : a \neq b\}.$$

**Theorem 1.** Let $A$ be defined as in Proposition (4.1). Then $T_A$ has the Gibbs property with respect to $\mu_A$, where $\mu_A$ is the measure $\mu$ restricted to $A$.

Proof: Combining the observations that $T_A$ has the Rényi property and that $T_A$ is eventually expanding, it immediately follows that $T_A$ has the Gibbs property with respect to $\lambda$ (for details see for instance [Aa], Proposition 4.3.3). Moreover, a straightforward calculation shows that $\log(d\mu / d\lambda)$ is a bounded, continuous function on $A$. Hence $\log(d\mu / d\lambda)$ is Lipschitz continuous on $A$ which implies by Proposition 4.7.1 in [Aa] that $T_A$ also has the Gibbs property with respect to the invariant measure $\mu$.

### 4.3 Ergodic properties of the coding map

By a result of Aaronson, Denker and Urbanski (see [ADU, Theorem 3.2] and [Aa, Theorem 4.4.7]), a topologically mixing, conservative Gibbs–Markov map is exact. Thus by Propositions 4.2, 4.4 and Theorem 1 we have that $T_A$ is exact (and hence ergodic). Since $T_A$ is conservative the map $T$ is exact (and ergodic) as well. For systems of this type there is a further classification (see [Aa]).

A conservative, ergodic, measure preserving transformation $T$ of $(X, \mathcal{B}, \mu)$ is called **rationally ergodic** if there is a set $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$ and a constant $M > 0$ such that

$$\int_A \left( \sum_{i=0}^{n-1} 1_A \circ T^i \right)^2 d\mu \leq M \left( \int_A \sum_{i=0}^{n-1} 1_A \circ T^i d\mu \right)^2 \quad \forall n \geq 1.$$  

(4)
Furthermore there is a sequence \((a_n)\), \(a_n \not\to \infty\) associated to \(T\) which satisfies the following. For a set \(A\) which satisfies Equation 4, we have that
\[
\frac{1}{a_n} \sum_{i=0}^{n-1} \mu(B \cap T^{-i}C) \to \infty \quad m(B)m(C) \quad \forall B, C \in \mathcal{B} \cap A.
\]
This sequence is unique up to asymptotic equality and is called the return sequence of \(T\).

A stronger ergodic property of \(T\) is defined via the transfer operator \(\hat{T} : L^1(\mu) \to L^1(\mu)\) which is given by
\[
\int_X \hat{T}f \cdot gd\mu = \int_X f \cdot g \circ Td\mu \quad \forall f \in L^1(\mu), g \in L^\infty(\mu).
\]
Namely, a conservative, ergodic, measure preserving transformation \(T\) is pointwise dual ergodic if there is a sequence \((b_n)\), \(b_n \not\to \infty\) such that
\[
\frac{1}{b_n} \sum_{i=0}^{n-1} \hat{T}^i f \to \int_X f d\mu \quad \text{a.e. as } n \to \infty \forall f \in L^1(X).
\]
Any pointwise dual ergodic transformation is rationally ergodic and the sequences \((a_n)\) and \((b_n)\) coincide up to asymptotic equality (see [Aa], Proposition 3.7.1). Therefore \((b_n)\) is also referred to as the return sequence of \(T\).

**Theorem 2.** Let \(G\) be a cofinite Fuchsian group which is not cocompact. Then the coding map \(T\) is pointwise dual ergodic with respect to \(\mu\). The return sequence \((a_n)\) of \(T\) is given by
\[
a_n = \frac{\pi}{4(\text{Area}(\mathbb{H}/G) + 2\pi)} \cdot \frac{n}{\log(n)}.
\]

**Proof:** Since \(T_A\) has the Gibbs-Markov property, Theorem 4.8.1 in [Aa] gives that \(T\) is pointwise dual ergodic and that \(A\) is a Darling–Kac set. That is, there exists a sequence \((a_n), a_n \not\to \infty\) such that
\[
\frac{1}{a_n} \sum_{i=0}^{n-1} \hat{T}^i 1_A \to \mu(A) \quad \text{almost uniformly on } A.
\]
Hence it follows by the Chacon–Ornstein theorem that \((a_n)\) is a return sequence for \(T\). Furthermore, since the wandering rate \(L_A(n)\) of \(A\) is proportional to \(\log(n)\) (Proposition 4.1), \(L_A(n)\) is regularly varying at \(\infty\) with index \(\alpha = 0\). Using Proposition 3.8.7 in [Aa], we obtain that
\[
a_n \sim \frac{1}{\Gamma(2-\alpha)\Gamma(1+\alpha)} \frac{n}{L_A(n)} \cdot \frac{1}{\log(n)} \cdot \frac{1}{4\#V}.
\]
Since \(\text{Area}(P) = (\#V - 2)\pi\), the return sequence is given by
\[
a_n = \frac{\pi}{4(\text{Area}(\mathbb{H}/G) + 2\pi)} \cdot \frac{n}{\log(n)}.
\]

**Corollary 4.5.** The first return map \(S\) is rationally ergodic with return sequence \((a_n)\).

**Proof:** By the last theorem, \(T\) is rationally ergodic. The assertion follows by the fact that \(S\) is the natural extension of \(T\) (see Proposition 3.2). We note that no invertible transformation can be pointwise dual ergodic.
5 A different choice of the fundamental domain

The construction of the coding map $T$ presented in here relies on the choice of a specially shaped fundamental polygon. Namely, for a cofinite Fuchsian group $G$ with parabolic elements, we use the fact that there exists a fundamental polygon $P$ such that $P$ is an ideal polygon, that is all vertices of $P$ are contained in $\partial \mathbb{H}$. This shape of $P$ then immediately gives that the section $(Y, S)$ has the Markov property and that the invariant measure induced by the Liouville measure is infinite. Moreover, the combinatorial structure of the canonical factor $T$ is less complicated than the one of the map introduced in [BS].

Recall that in [BS], the construction of the coding map $T$ relies on the choice of fundamental polygon which satisfies the so called even corner property or net condition. Namely, a fundamental domain $P$ for a given group $G$ satisfies the even corner property if $G(\partial P)$ is the union of geodesics. Note that such $P$ exists for any cofinite Fuchsian group $G$ (see [BS]), and that the class of ideal polygons is included in the class of fundamental polygons with the latter property.

Moreover, by a result obtained in [Se2] we have that the first return map of the Poincaré section $\{(\xi, \eta) : \exists \xi \in \mathbb{R} \text{ s.t. } \gamma_{\xi, \eta}(t) \in P\}$ is conjugated to a Markov map $S$ acting on some subset $Y$ of $\partial \mathbb{H} \times \partial \mathbb{H}$. Combining the fact that the conjugating map is piecewise defined via elements of $G$ and that the measure $\mu$ given by $d\mu(\xi, \eta) = (d|\xi||\eta|)/|\xi - \eta|^2$ is invariant under the action of $\text{Iso}^+(\mathbb{H})$, one clearly obtains that $(Y, S, \mu, T)$ is a measure preserving Markov map. Due to the construction of $Y$ it follows immediately that $\mu(Y)$ is infinite if and only if there are some vertices of $P$ contained in $\partial \mathbb{H}$ which is equivalent to the non-compactness of $\mathbb{H}/G$.

Following [BS], [Se1] and [AF] one obtains that the canonical non–invertible factor $(\partial \mathbb{H}, B, \mu, T)$ is a topological mixing, measure preserving Markov map where the measure $\mu$ is the image measure of the factor map as in Proposition 3.1. Recall that this factor is defined as follows.

Let the sides $S = \{s_1, s_2, \ldots, s_k\}$ of $P$ be labelled in anticlockwise order and denote by $\gamma(i)$, for $s_i \in S$, the geodesic containing $s_i$. Furthermore, let $P_i, Q_i$ be the elements of $\partial \mathbb{H}$ such that $\{P_i, Q_i\}$ is the set of endpoints of $a(\gamma(i))$ and such that $P_i$ comes before $Q_i$ in anticlockwise order. Now $T : \partial \mathbb{H} \rightarrow \partial \mathbb{H}$ is partially defined as follows (see figure 3).

$$x \mapsto g_{s_i}(x) \text{ for } x \in [P_i, P_{i+1}] \text{ for } i = 1, \ldots, k-1 \text{ and } x \mapsto g_{s_k}(x) \text{ for } x \in [P_k, P_1].$$

We are now in position to adapt the analysis presented in here to this situation. Therefore, observe the following. For $s_i \in S$ such that the endpoints of $s_i$ are contained in $\mathbb{H}$ we have that $g_i(P_{j-1})$ and $g_i(Q_{j+1})$ are contained in the interval $[Q_{j-1}, P_{j+1}]$ where $j$ is given by $g_i(s_i) = s_j$ and the indices are considered to be taken modulo $k = S$ (see figure 3). For a cocompact group $G$ the proof of Lemma 4.3 can now be easily adapted as follows.

Namely, if $T^n[\omega] = g[\omega]$ for some interval $[\omega] \subset \partial \mathbb{H}$ and $g \in G$ such that $[\omega] \subset T^n([\omega])$ we have by the latter observation that the repelling fixed point $m_{\omega-1}$ of $g^{-1}$ is contained in $[Q_{j-1}, P_{j+1}]$ for some suitable $j \in \{1, 2, \ldots k\}$. Hence, for $\epsilon := \min\{[Q_{j-1}, P_{j+1}] : j \in \{1, 2, \ldots k\}\}$, we obtain by the same arguments as in the first case of the proof of Lemma 4.3 that $|D^2 g(z)/|D g(z)|^2| < 4/\epsilon$ for all $z \in [\omega]$. If $[\omega] \not\subset T^n([\omega])$, a similar argument applies.

Following the chain of arguments of sections 4.2 and 4.3 we deduce the following well known result (see [BS], [AF]).

**Theorem 3.** Let $G$ be a cocompact Fuchsian group. Then the map $T$ is an ergodic, measure preserving Gibbs–Markov map with respect to the finite measure $\mu$.

Note that in this case the finiteness and invariance of $\mu$ immediately give that $T$ is conservative. So assume from now on that $\mu$ is infinite or equivalently that $G$ is not
cocompact. In this situation the property that $T$ is conservative can be deduced as in Proposition 4.2 by the existence of a set $A$ of finite measure such that $\bigcup_{i=1}^{\infty} T^{-i}(A) = \partial H$ mod $\mu$. Moreover, the wandering rate can be specified precisely as in Proposition 4.1. Namely, for $A$ bounded away from the set of ideal vertices $V^* \subset \partial H$ of $P$ we have that
\[
\lim_{n \to \infty} \frac{\mu\left(\bigcup_{i=0}^{n} T^{-i}A\right)}{\log n} = 4 \# V^*.
\]
By using the above observation the analogue of Lemma 4.3 can be easily obtained in this situation. Hence, by the arguments of sections 4.2 and 4.3, we obtain the following analogue of Theorem 2.

**Theorem 4.** Let $G$ be a cofinite Fuchsian group which is not cocompact. Then the map $T$ is an ergodic, conservative, measure preserving Markov map with respect to the infinite measure $\mu$. Moreover, $T$ is pointwise dual ergodic and the associated return sequence is given by
\[
a_n = \frac{n}{\log(n)} \cdot \frac{1}{4\# V^*}.
\]

**References**


