(2) Teorema da Média e o Principio do Maximo para funções harmonicas.

Def: $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is called harmonic if
(Ω open)

1) $u \in C^2(\Omega)$

2) $\Delta u = 0$, where Δ is Laplaciano:

$$\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

Examples: 1) constante

2) $u = a + \sum_{i=1}^{n} b_i x_i + \sum_{i \neq j} c_{ij} x_i x_j$

3) fundamental solution:

we look for $u = u(r)$ solution of $\Delta u = 0$

spherical coordinates $\Rightarrow$ $\frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial u}{\partial r} = 0$

$$r^{n-1} \frac{\partial u}{\partial r} = c_1$$

$$u = \begin{cases} c_1 \ln |r| + c_2, & n = 2 \\ c_1 r^{2-n} + c_2, & n \geq 3 \end{cases}$$

Def: $$\mathcal{E}(x) = \begin{cases} \frac{1}{2\pi} \ln |x|, & n = 2 \\ \frac{1}{(2-n)|S_1|} |x|^{2-n}, & n \geq 3 \end{cases}$$

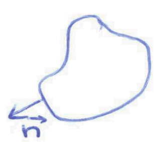
$|S_1|$ is the area of unit sphere in $\mathbb{R}^n$

4) real and imaginary part of complex analytic functions $(\varphi + i\psi)$ due to Cauchy-Riemann equalities

Useful formula: (Green) $v, u \in C^2(\Omega)$, Ω-open, $\partial\Omega \in C^2$ (smooth enough to have normal)

$$\int_\Omega (\Delta u \cdot v - u \cdot \Delta v)\, dx = \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \cdot v - u \cdot \frac{\partial v}{\partial n} \right) dS(x)$$

Here $\frac{\partial u}{\partial n} = \nabla u \cdot n$ - normal derivative



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! I know that I have somewhere error in signs ($\pm$), but do not have enough time to find where exactly it is. Please see modulo $\pm$.

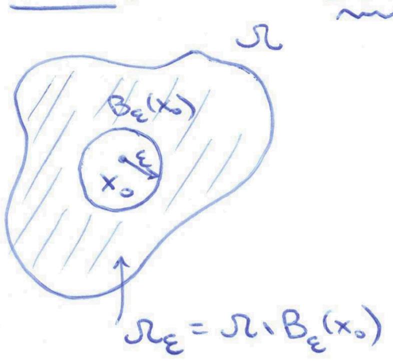
<u>Proposition 1</u>: (representation of a C^2-function in terms of potentials)

$u \in C^2(\Omega)$, $x_0 \in \Omega$

Then

$$u(x_0) = -\int_\Omega \Delta u(y)\, \mathcal{E}(x_0-y)\,dy + \int_{\partial\Omega} \frac{\partial u}{\partial n}(y)\, \mathcal{E}(x_0-y)\, dS(y) \mp \int_{\partial\Omega} \frac{\partial \mathcal{E}}{\partial n}(x_0-y)\,u(y)\, dS(y)$$

<u>Proof</u>: Idea: use Green formula for u and $v = \mathcal{E}$ - fundamental solution in $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(x_0)$.



We need to exclude $B_\varepsilon(x_0)$ to use the fact that $\Delta\, \mathcal{E}(x_0-y) = 0$.

$$\int_{\Omega_\varepsilon} \left(\Delta_y u(y)\, \mathcal{E}(x_0-y) - u(y)\, \Delta_y \mathcal{E}(x_0-y)\right) dy = \int_{\partial\Omega_\varepsilon} \frac{\partial u}{\partial n}(y)\, \mathcal{E}(x_0-y)\, dS_y - \int_{\partial\Omega_\varepsilon} u(y) \frac{\partial \mathcal{E}}{\partial n}(x_0-y)\, dS(y)$$

First, we state, then prove:

1) $\Delta_y \mathcal{E} = 0$ in Ω_ε [by def. of fund. solution]

2) $\displaystyle\int_{\partial B_\varepsilon(x_0)} \frac{\partial u}{\partial n}(y) \cdot \mathcal{E}(x_0-y)\, dS(y) \xrightarrow[\varepsilon\to 0]{} 0$

3) $\displaystyle\int_{\partial B_\varepsilon(x_0)} u(y) \frac{\partial \mathcal{E}}{\partial n}(x_0-y)\, dS(y) \to u(x_0)$

From these three statements the formula in Prop. 1 follows.

Proof of 2): $\displaystyle\int_{S_\varepsilon(x_0)} \frac{\partial u}{\partial n}(y) \cdot \frac{\text{const}}{|x_0-y|^{n-2}}\, dS(y) = \Big/$ change of variables to ~~spherical~~ $x_0 - y = x_0 - \varepsilon z$ $\Big/$

$S_\varepsilon(x_0) = \partial B_\varepsilon(x_0)$

$$= \int_{S_1(x_0)} \frac{\partial u}{\partial n}(\varepsilon z) \cdot \text{const} \cdot \underbrace{\frac{\varepsilon^{n-1}}{\varepsilon^{n-2}}}_{\sim \varepsilon}\, dS(z) \sim c \cdot \varepsilon \to 0$$

as $\left|\frac{\partial u}{\partial n}\right| \le M$ $(u \in C^2)$

Proof of 3): $\frac{\partial \mathcal{E}(x_0-y)}{\partial n} = \frac{1}{|S_1|(2-n)} \left(\frac{1}{r^{n-2}}\right)'_r \Big|_{r=x_0-y} = \frac{1}{|S_1|} \frac{1}{|x_0-y|^{n-1}}$

$$\int_{S_\varepsilon(x_0)} u(y) \cdot \frac{1}{|S_1|} \underbrace{\frac{1}{|x_0-y|^{n-1}}}_{\varepsilon} dS(y) = \underbrace{⨍_{S_\varepsilon(x_0)} u(y)\, dS(y)}_{\text{mean over a sphere } S_\varepsilon(x_0)} \xrightarrow[\varepsilon \to 0]{} u(x_0) \text{ as } u \text{ is cont.}$$

Q.E.D.

$\boxed{\text{Thm}}$ (mean-value theorem for harmonic functions):

$u: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ – harmonic

Then $\forall x_0 \in \Omega$, $\forall r > 0$ $\quad u(x_0) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(y)\, dy$

(or equivalently $u(x_0) = \frac{1}{|S_r(x_0)|} \int_{S_r(x_0)} u(y)\, dy$

Comment: to check the equivalency just write

$$u(x_0) \cdot |B_1(x_0)| \cdot r^n = \int_{B_r(x_0)} u(y)\, dy = \int_0^r \int_{S_r(x_0)} \dots$$

and differentiate in r. The same for sphere but instead of differentiation integrate.

Proof:

Idea: use Prop. 1 for $\Omega = B_r(x_0)$. We have

$$u(x_0) = -\int_{B_r(x_0)} \underbrace{\Delta u(y)}_{=0} \mathcal{E}(x_0-y)\, dy + \int_{S_r(x_0)} \frac{\partial u}{\partial n}(y)\, \mathcal{E}(x_0-y)\, dS(y) - \int_{S_r(x_0)} \frac{\partial \mathcal{E}}{\partial n}(x_0-y)\, u(y)\, dS(y)$$

as u is harmonic

We have:

1) $\int_{S_r(x_0)} \frac{\partial u}{\partial n}(y)\, \mathcal{E}(x_0-y)\, dy = 0$ and 2) $\int_{S_r(x_0)} \frac{\partial \mathcal{E}}{\partial n}(x_0-y)\, u(y)\, dy = ⨍_{S_r} u(y)\, dS(y)$

Proof of 1) :

$$\int_{S_r(x_0)} \frac{\partial u}{\partial n}(y) \cdot \underbrace{\frac{1}{(2-n)|S_1|} \frac{1}{\varepsilon^{n-2}}}_{\text{constant}} dS(y) = C_1 \cdot \int_{S_r(x_0)} \frac{\partial u}{\partial n}(y)\, dy$$

Again use Green formula for u and $v \equiv 1$. We obtain

$$\int_{B_r(x_0)} (\underset{0}{\Delta u} \cdot 1 - u \cdot \underset{0}{\Delta 1})\, dy = \int_{S_r(x_0)} \left(\frac{\partial u}{\partial n}(y) \cdot 1 - u \cdot \underset{0}{\frac{\partial 1}{\partial n}}\right) dS(y) \Rightarrow \int_{S_r(x_0)} \frac{\partial u}{\partial n} = 0$$

Proof of 2) $\int_{S_r(x_0)} u(y) \cdot \frac{\partial \varepsilon(x_0-y)}{\partial n}\, dS(y) = \int_{S_r(x_0)} u(y) \cdot \frac{1}{|S_1|} \frac{1}{r^{n-1}}\, dS(y) =$

$$= \fint_{S_r(x_0)} u(y)\, dy \quad \text{by def.}$$

Q.E.D.

It turnes out that the inverse statement is also true.

$\boxed{\text{Thm}}$ (Inverse of mean-value theorem)

$\Omega \subseteq \mathbb{R}^n$ - open

Let $\forall x_0 \in \Omega$ $\quad u(x_0) = \fint_{S_r(x_0)} u(y)\, dy \Rightarrow$ 1) $u \in C^\infty(\Omega)$
2) $\Delta u = 0$

Proof:

Let us first prove 2). As we have already seen, for $\Omega = B_r(x_0)$ the Green formula for u and $v = \varepsilon$ gives:

$$-u(x_0) = \int_{B_r(x_0)} \Delta u \cdot \varepsilon + \int_{S_r(x_0)} \frac{\partial u}{\partial n}(y) \frac{1}{|S_1|(2-n)} \frac{1}{r^{n-2}}\, dS(y) - \fint_{S_r(x_0)} u(y)\, dy$$

The terms in ◯ coincide by assumption of the Theorem

Thus, we have (see next

$$\int_{B_r(x_0)} \Delta u \cdot \varepsilon + \int_{S_r(x_0)} \frac{\partial u}{\partial n}(y) \frac{1}{|S_1|(2-n)} \frac{1}{r^{n-2}} \, dS(y) = 0$$

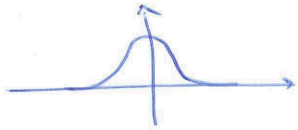
$$\parallel$$

$$+ \frac{1}{|S_1|(2-n) r^{n-2}} \int_{B_r(x_0)} \Delta u(y) \, dS(y)$$

$$\Rightarrow \int_{B_r(x_0)} \Delta u \underbrace{\left(\frac{1}{|x_0-y|^{n-2}} - \frac{1}{r^{n-2}} \right)}_{\geq 0} dS(y) \cdot \underbrace{\frac{1}{|S_1|(2-n)}}_{\text{constant}} = 0$$

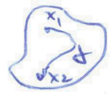
$$\Rightarrow \Delta u \equiv 0 .$$

Proof of 1) Let us prove that $u \in C^\infty(\Omega)$.
One can use mollifiers

$\eta(x) =$  $\eta \in C^\infty(\mathbb{R})$ supp $\eta = [-1,1]$
$x \in \mathbb{R}$

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{\|x\|}{\varepsilon}\right), \quad x \in \mathbb{R}^n$$

$u_\varepsilon = u * \eta_\varepsilon$ is C^∞ and $u_\varepsilon \to u$
↑ convolution



$\boxed{\text{Thm}}$ (Maximum principle)

Let $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be harmonic

Ω - open, connected ($\forall x_1, x_2 \in \Omega$ $\exists$ curve $\gamma : [a,b] \to \mathbb{R}^n$
$\gamma(a) = x_1$, $\gamma(b) = x_2$, $\gamma \subset \Omega$)

Let $\underset{x \in \Omega}{\arg\max}\, u = x_0 \in \underset{\Omega}{\overset{\shortparallel}{\operatorname{Int}(\Omega)}} \Rightarrow u \equiv \text{const}$

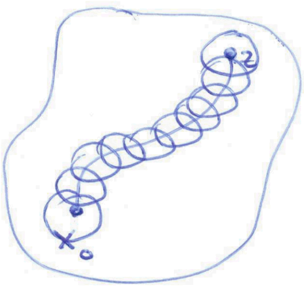
Corollary: $\max\limits_{x \in \overline{\Omega}} u(x) = \max\limits_{x \in \partial\Omega} u(x)$. Maximum is attained at $\partial\Omega$

The same with minimum.

Proof: use mean-value theorem

$$u(x_0) = \fint_{B_r(x_0)} u(y)\,dy \iff 0 = \fint_{B_r(x_0)} \underbrace{(u(x_0) - u(y))}_{\geq 0} dy$$

$\Rightarrow u(y) \equiv u(x_0)$ for $y \in B_r(x_0)$.

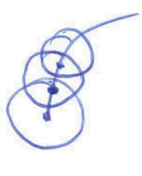


Take any $z \in \Omega$ and a curve $\gamma : [a,b] \to \mathbb{R}^n$ s.t. $\gamma(a) = x_0$
$\gamma(b) = z$

Cover $\gamma[a,b]$ with finite number of balls (it is possible as $\gamma[a,b]$ is a compact set)

Then in all balls the value equals to $u(x_0)$

Thus $u(z) = u(x_0) \Rightarrow u \equiv u(x_0)$ constant



Q.E.D.

Applications:

Thm (uniqueness of solutions to Dirichlet problem of Poisson equation)

$\exists\, u \in C^2(\Omega)$:
$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases} \Rightarrow \text{If solution exists it is } \underline{\text{unique}}$$

Proof: Take u_1, u_2 – two solutions.

$\not\exists\; v = u_1 - u_2 \Rightarrow \begin{cases} \Delta v = 0 & \text{in } \Omega \\ v|_{\partial\Omega} = 0 \end{cases}$

$$\Rightarrow \min_{\bar\Omega} v = \max_{\bar\Omega} v = 0 \Rightarrow v \equiv 0 \Rightarrow u_1 \equiv u_2$$

Q.E.D.

Max principle can be made more "strict"

Thm u - harmonic, $\partial\Omega$ is smooth enough

$\exists\, x_0 = \underset{x \in \bar\Omega}{\operatorname{argmax}}\, u(x) \in \partial\Omega$

Then $\frac{\partial u}{\partial n}(x_0) > 0$.

(without proof, but idea is to use barrier functions)

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1) Inverse function theorem, Implicit function theorem and applications.

Let $\begin{cases} y_1 = f_1(x_1 \dots x_n) \\ \vdots \\ y_n = f_n(x_1 \dots x_n) \end{cases}$ (1) We would like to solve the inverse problem: $\begin{cases} x_1 = g_1(y_1 \dots y_n) \\ \vdots \\ x_n = g_n(y_1 \dots y_n) \end{cases}$

When is it possible?

Idea: if we look at linear eqs $y = Ax$, $x, y \in \mathbb{R}^n$, A - matrix $n \times n$

then $x = A^{-1}y$ if A is invertible
($\exists A^{-1}$ or $\det A \neq 0$ or rank $A = n$)

In the vicinity of every point $(x_1^0 \dots x_n^0)$ the relation (1) is approximated by a linear operator

$$y = DF(x^0) \cdot x, \text{ where } DF = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \vdots \\ \vdots & \ddots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

So we guess that if
$DF(x^0) \neq 0$ then (at least locally) we have inverse

This is called inverse function theorem.

$\boxed{\text{Thm}}$ (Inverse Function Theorem) $f: \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ ($\mathcal{U}$ open), $x_0 \in \mathcal{U}$:
- $f \in C^1(\mathcal{U})$
- $Df(x_0)$ is invertible

$f(x_0) = y_0$

Then: 1) $\exists\, U_1, V_1 \subset \mathbb{R}^n$ - neighborhoods of x_0 and y_0 resp.
s.t. $f(U_1) = V_1$ and f is bijection
($\Rightarrow \exists f^{-1} =: g : V_1 \to U_1$)

2) $g \in C^1(V_1)$

Remark: condition $f \in C^1(\mathcal{U})$ is important.
Counterexample $f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$f'(0) = \frac{1}{2}$, but f is not locally invertible

Proof:

Take λ : $2\lambda \cdot \|A^{-1}\| = 1$

and $x \in \mathcal{U}$ s.t. $\| f'(x) - A \| < \lambda$

$\varphi(x) = x + A^{-1}(y - f(x))$ for some $y \in \mathbb{R}^n$

Note that $\varphi(x) = x \iff y = f(x)$

So if for any fixed y we prove that φ has a unique fixed point, we prove that $y = f(x)$ has unique solution x.

To prove it, we need the following auxiliary notions and statements:

Def: $\varphi : M \to M$ is a contraction if $\exists K \underset{\geq 0}{<} 1$

$$d(\varphi(x_1), \varphi(x_2)) \leq K\, d(x_1, x_2) \quad \forall x_1, x_2 \in M$$

for M being a complete metric space

Thm (Contraction ~~point thm~~ maps theorem)

$\varphi : M \to M$ - contraction $\Rightarrow \exists!$ fixed point $x \in M$: $\varphi(x) = x$.

M - complete metric space

Continuation of the proof. Plan:

1. φ is a contraction
2. $\varphi : B_r(x_0) \to \underbrace{B_r(x_0)}_{\text{complete metric space}}$

~~3.~~ $1 + 2 \Rightarrow \varphi$ has a unique fixed point by contraction maps theorem $\Rightarrow \exists f^{-1} =: g$

3. $\exists U_1, V_1$ $f(U_1) = V_1$ is open and $g : V_1 \to U_1$ is well-defined
4. $g \in C^1(V_1)$

• Proof of 1)

$$\varphi'(x) = 1 - A^{-1} f'(x) = A^{-1}A - A^{-1} f'(x) = A^{-1}(A - f'(x))$$

$$\|\varphi'(x)\| \leq \|A^{-1}\| \cdot \|A - f'(x)\| \leq \frac{1}{2\lambda} \cdot \lambda = \frac{1}{2}$$

$$\Rightarrow \|\varphi(x_1) - \varphi(x_2)\| \leq \|\varphi'(x)\| \cdot \|x_1 - x_2\| \leq \frac{1}{2}\|x_1 - x_2\|.$$

• Proof of 2). Take $x \in B_r(x_0)$.

$$\|\varphi(x) - x_0\| \leq \underbrace{\|\varphi(x) - \varphi(x_0)\|}_{\leq \frac{1}{2}\|x - x_0\|} + \underbrace{\|\varphi(x_0) - x_0\|}_{\leq \|A^{-1}\| \cdot \|y - f(x_0)\| \leq \frac{1}{2\lambda} \cdot \|x - x_0\| \cdot \lambda} \leq \|x - x_0\| \leq r$$

at least for $y \in B_{\lambda r}(f(x_0))$.

Thus by contraction map theorem $\exists! x: \varphi(x) = x \Rightarrow \forall y \in B_{\lambda r}(f(x_0))$ $\exists! x \in \mathcal{U}: y = f(x)$

• Take $V_1 = B_{\lambda r}(f(x_0)) \setminus S_{\lambda r}(f(x_0))$ – open ball

As we have seen above for all $y \in V_1$ $\exists! x \in \mathcal{U}$

thus we can define $f^{-1} = g : x \mapsto y$

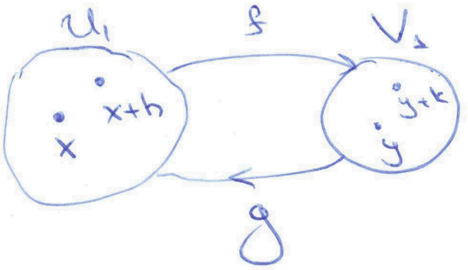
$$g : V_1 \to g(V_1)$$

clear that it is open

Proof of 4) To prove this we will prove that $\exists T$:

$$\frac{\|g(y+k) - g(y) - Tk\|}{\|k\|} \leq \text{const} \cdot \frac{\|f(x+h) - f(x) - Df(x) \cdot h\|}{\|h\|} \to 0$$

as $\|h\| \to 0$ $\Rightarrow$ $\boxed{\begin{array}{l} LHS \to 0 \\ \text{as } \|k\| \to 0 \end{array}}$



• $\|k\| \leq \dfrac{\|h\|}{\lambda}$

Thm (Implicit Function Theorem)

$F: \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m$, $F \in C^1$

$\exists\ F(x_0, y_0) = 0$, $x_0 \in \mathbb{R}^k$
$y_0 \in \mathbb{R}^m$

$\exists\ \frac{\partial F}{\partial y}(x_0, y_0)$ is invertible, then

$\exists\ W \subset \mathbb{R}^k$ and function $\varphi: W \to \mathbb{R}^m$:

1) $\varphi(x_0) = y_0$

2) $F(x, \varphi(x)) \equiv 0 \quad \forall\ x \in W$.

Proof:

Consider a map: $G: \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k \times \mathbb{R}^m$
$(x, y) \mapsto (x, F(x,y))$

For this function we can use Inverse Function Theorem. Indeed

$$DG\Big|_{(x_0,y_0)} = \begin{pmatrix} I_k & 0 \\ 0 & D_y F \end{pmatrix}\Big|_{(x_0,y_0)} \Rightarrow \det DG\Big|_{(x_0,y_0)} = \det D_y F \neq 0$$

So there exists an inverse function

$(x, z) \mapsto (x, G(x, z))$

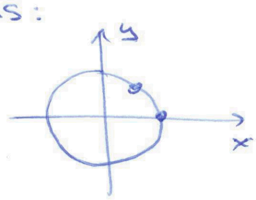
The first coordinate does not change $(x \mapsto x)$

We are looking for $F(x_0, y_0) = 0$, thus the desired function $\varphi(x) := G(x, 0)$

Q.E.D.

Aplications:

example:



$x^2+y^2=1$ $\quad F(x,y) = x^2+y^2-1=0$

At every point $(x_0,y_0) \neq (\pm 1, 0)$

$$\frac{\partial F}{\partial y} = 2y_0 \neq 0$$

$$\Rightarrow \exists^{\text{inverse}} \; y = \sqrt{1-x^2}, \text{ for } y>0$$

$$y = -\sqrt{1-x^2} \text{ for } y<0.$$

For $(\pm 1, 0)$ there does not exist an inverse.

(3) Fundamental theorem of Galois and applications.
Galois theory was invented to answer the following
questions: Sorry for unstructured text, only some ideas,

1) Can one solve polynomial eqs of type

$$x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = 0$$

using only "+", "−", ":", "x" and radicals?

We know explicit answers for $n = 2,3,4$ ("yes", using radicals, may be $\sqrt{a + \sqrt{b} + \sqrt{c}}$)
Galois theory can prove negative answer for $n=5$ and for any concrete eq. give an algorithm how to understand the answer.

2) Classical geometric questions.

- Can one using only "a ruler" and " " (to draw circles) construct a ~~circle of the~~ square of the same area as a given circle? (No)
- The same question for "doubling of the edge of a cube in $\mathbb{R}^3$" (No)
- The same question for constructing an angle $\frac{\alpha}{3}$ for a given angle α. (No)

The main concepts are;
Def. K-field, E - extention of a field if E is a field and $K \subseteq E$

Galois study possible extensions of the fields K constructing a sequence of extentions $K \subseteq E_1 \subseteq E_2 \subseteq \dots \subseteq E_n$ and give one-to-one ~~connect them~~ correspondance to some objects in group theory, namely group of automorphisms of extention and their subgroups k - field

K[x] - space of polynomials with coefficients in k

$p \in K[x]$, of order n.

If $\exists \alpha \in K$: $p(\alpha) = 0$, then $p(x) = (x-\alpha)\cdot p_1(x)$, $p_1 \in K[x]$

We know that if K is algebraically closed then $p(x)$ can be decomposed in a product of linear functions:

(1) $p(x) = (x-\alpha_1)^{d_1} \cdot \ldots \cdot (x-\alpha_k)^{d_k}$, $d_1 + \ldots + d_k = n$.

~~But in gener~~ This is an analogue of number theory when $a \in \mathbb{R}$ can be presented as a product of prime numbers.

The analogue of prime numbers is the "primitive polynomials"

But if k is not algebraically closed, the decomposition (1) will be with higher order polynomials.

The idea is that we can extend the field K and include some roots.

Example: $p(x) = x^2 - 2$, $k = \mathbb{Q}$, $\sqrt{2} \notin \mathbb{Q}$

But we can consider $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2}, a, b \in \mathbb{Q}\}$

This is a field and in this field $p(x) = (x-\sqrt{2})(x+\sqrt{2})$ can be decomposed.

Def: let α be a root of some polynomial $p(x)$.

Take $p(x)$ with minimal possible order.

$K(\alpha) := K[x]/p(x)$ - simple extention of K.

Note: $K(\alpha)$ is a vector space over K ~~with~~ and

~~and~~ also is the minimal field that contains both α and K.

The fields of decomposition of irreducible polynomials (that can not be decomposed in K) play import role in Galois theory.

Another object is $Aut_K(E)$ - group of all automorphisms j of extension $K \subseteq E$ such that

$j(x) = x \quad \forall x \in K$. ($K$ is fixed) not moving under j

<u>Properties</u>: 1. $j \in Aut_K(E)$

If α : $p(\alpha) = 0$ root of some polynomial, then $j(\alpha)$ is again a root. of the same polynomial.

2. $|Aut_K(E)| \leq n!$ $\quad$ $Aut_K(E)$ is a subgroup of S_n (group of permutations)

$K \subseteq E$ - extention of K

Take $f \in Aut_K(E)$. Consider $E_1 = \{a \in E : f(a) = a\}$

(E_1 is not changing under f)

So there is a correspondance of some extentions $E \longleftrightarrow G \subseteq Aut_K(E)$ - subgroup of automorphisms that do ~~a~~ not change E.

This correspondance is not always bijective

Galois characterized all situations when this is

bijective (homomorphisms - preserve operations too)

$k \subseteq E_1 \subseteq E$, $k \subseteq E_2 \subseteq E$

If E_1, E_2 are two extentions $E_1 \cap E_2$ is also an extention.

If $G_1 G_2$ are ~~two~~ subgroups of $Aut_k(E)$, then $G_1 \cap G_2$ is also subgroup.

Moreover, the correspondance is still valid

$$E_1 \cap E_2 \longleftrightarrow G_1 \cap G_2$$

Galois introduced the notion of extention of Galois ~~They correspond to~~ and proved that ~~H~~ every corres-ponding subgroup of $Aut_k(E)$ ~~should be~~ is (разрешима) ... sorry for Russian